Fitting Surfaces to Polygonal Meshes using Parametric Pseudo-Manifolds

Tutorial 3

SIBGRAPI 2008

XXI BRAZILIAN SYMPOSIUM ON COMPUTER GRAPHICS AND IMAGE PROCESSING

CAMPO GRANDE/MS - BRAZIL

October 12-15, 2008

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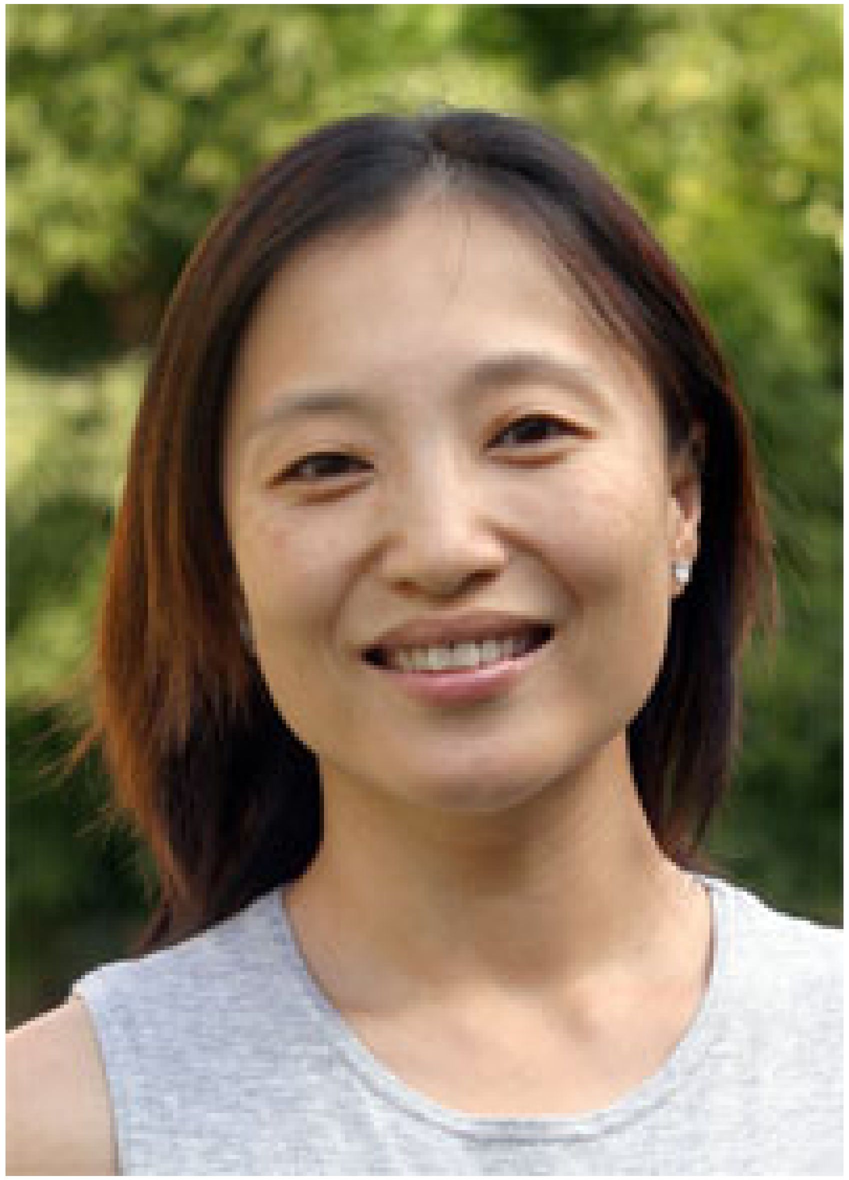

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Introduction

Marcelo Siqueira
UFMS

Outline

- The Surface Fitting Problem
- Traditional Approaches
- The Manifold-Based Approach
- What's Next?

The Surface Fitting Problem

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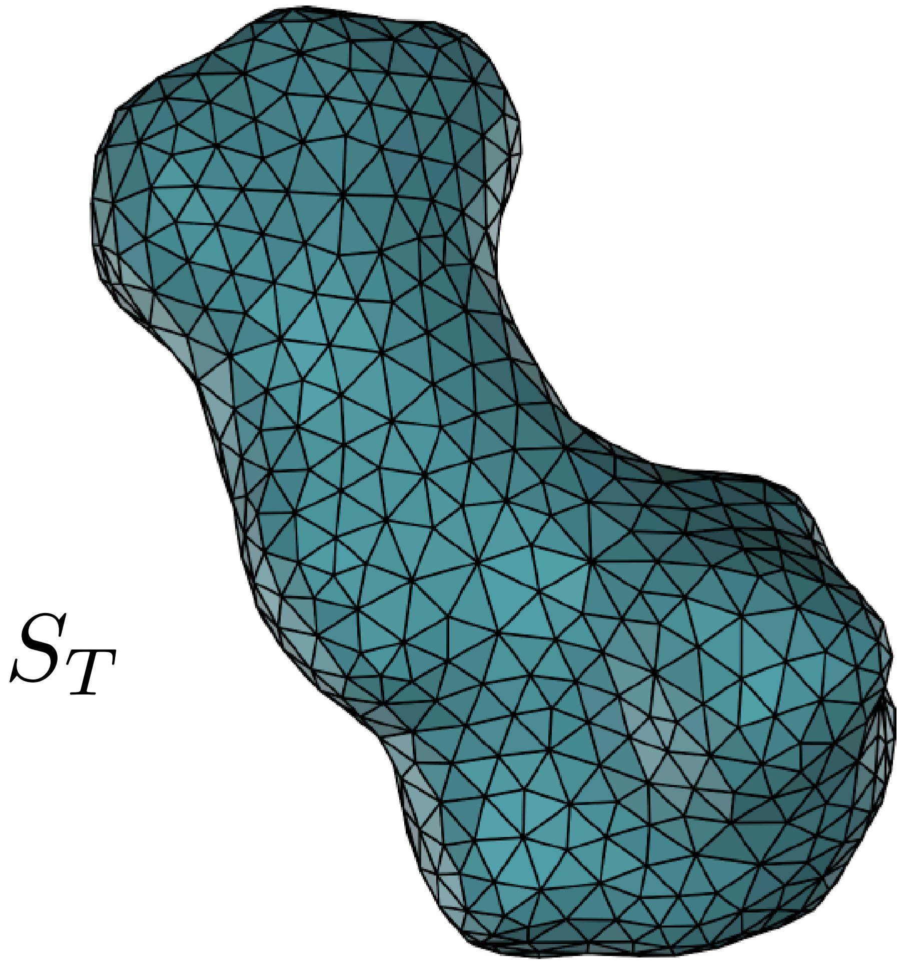

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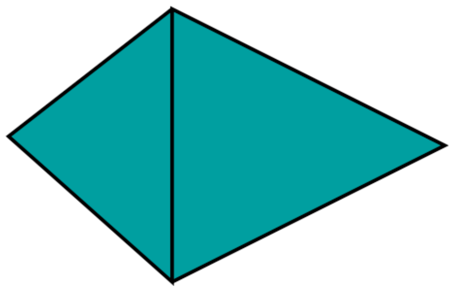
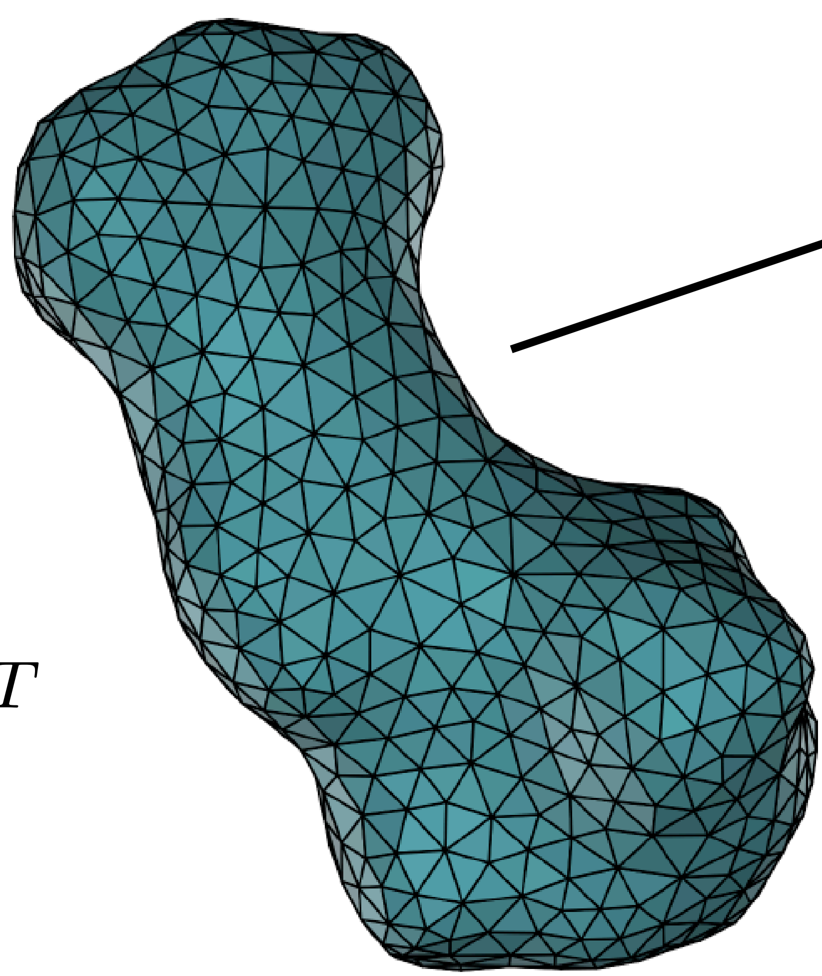


Edge property

S_T



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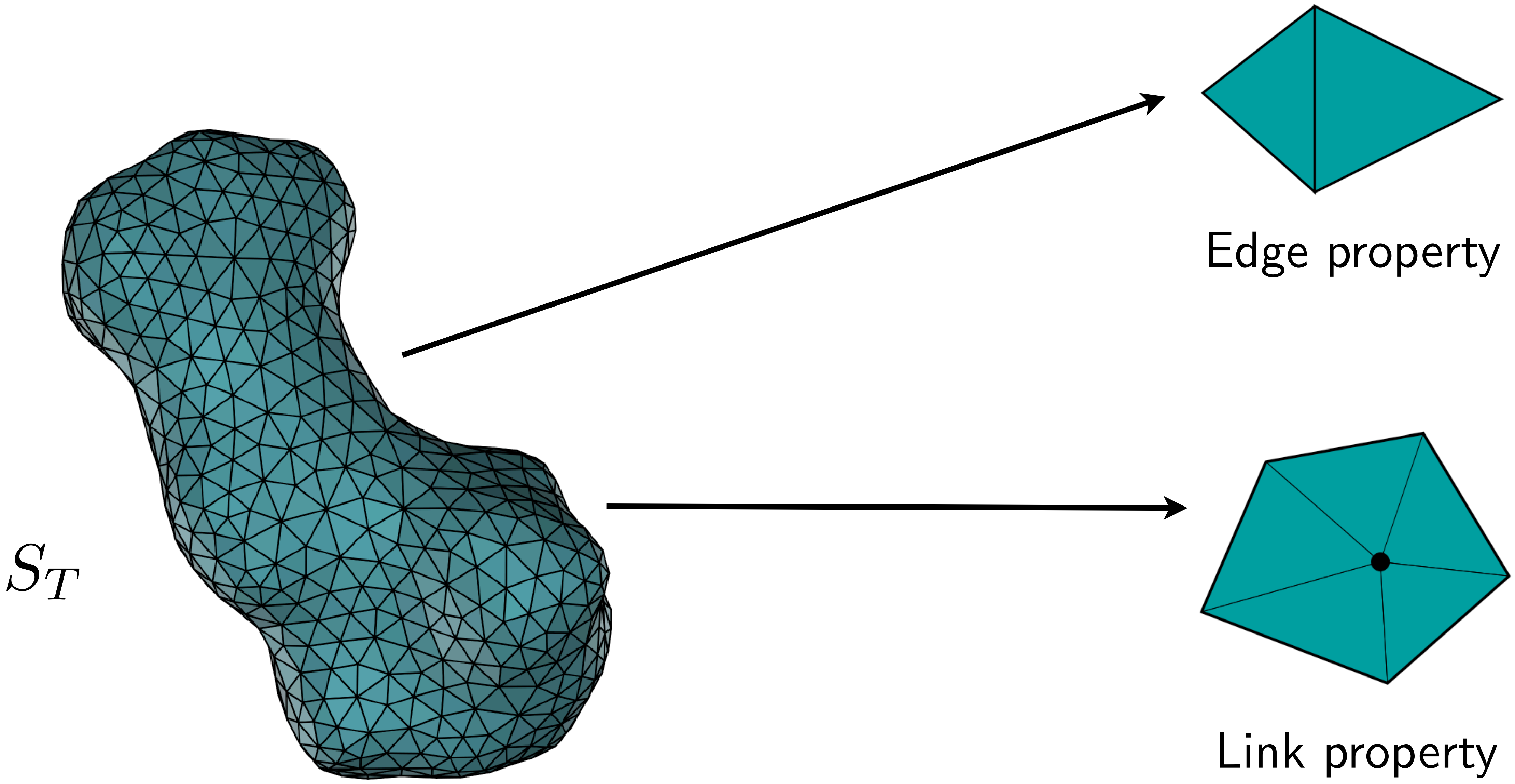

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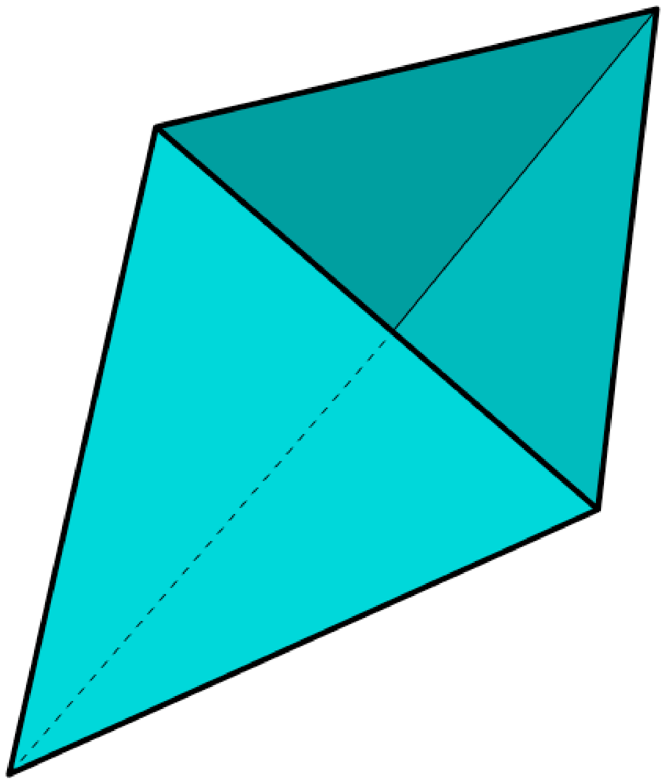

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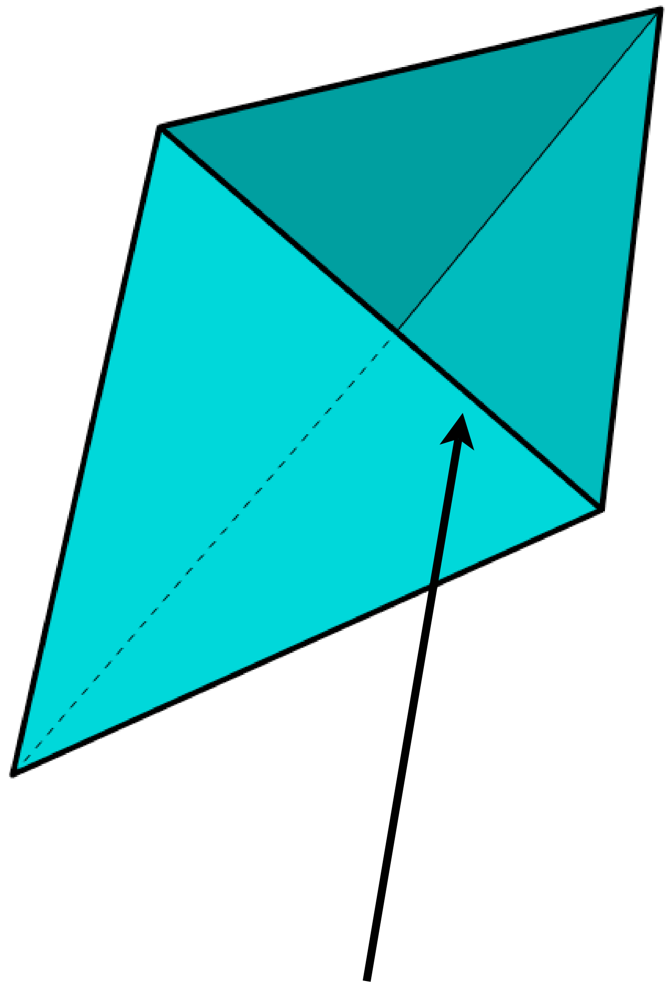

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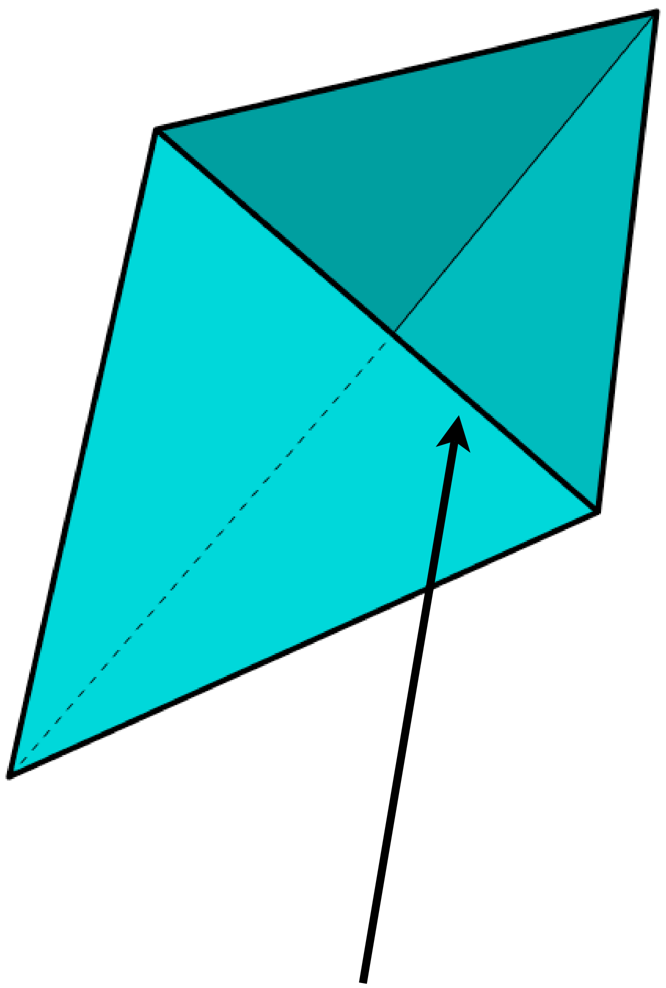

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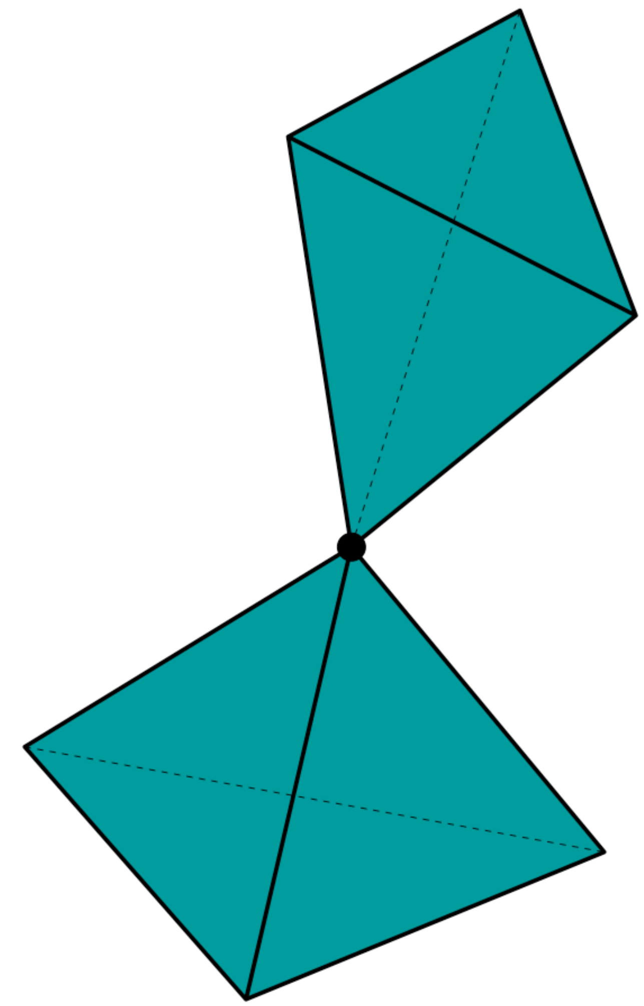

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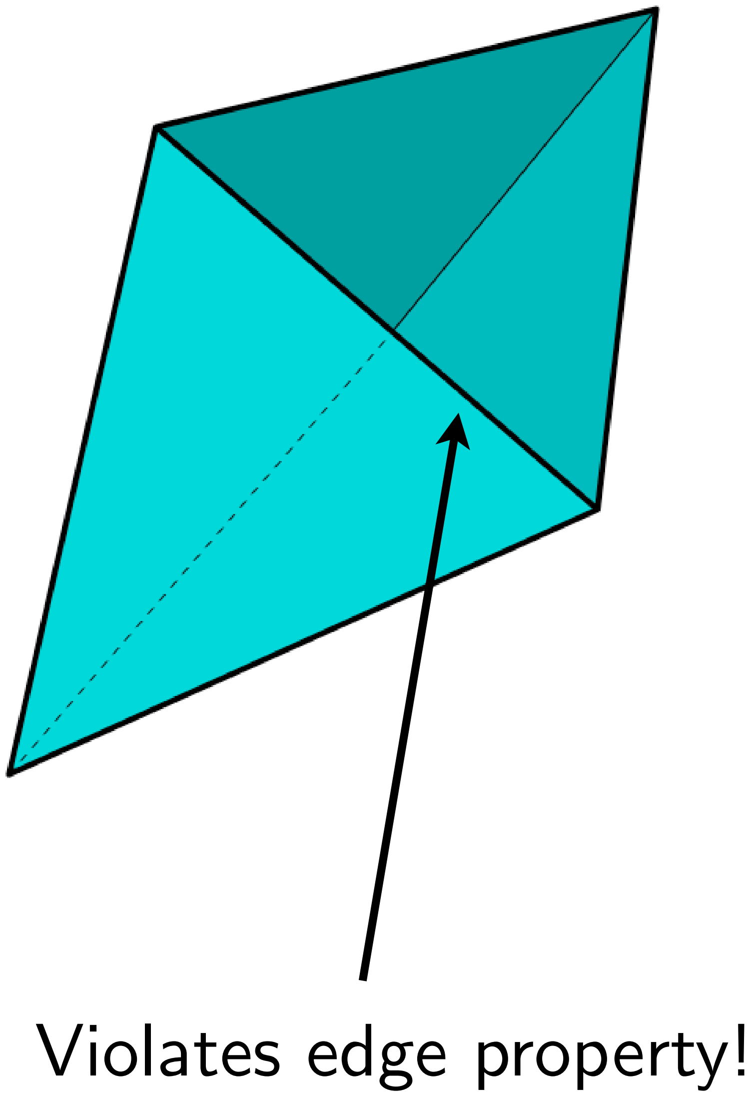

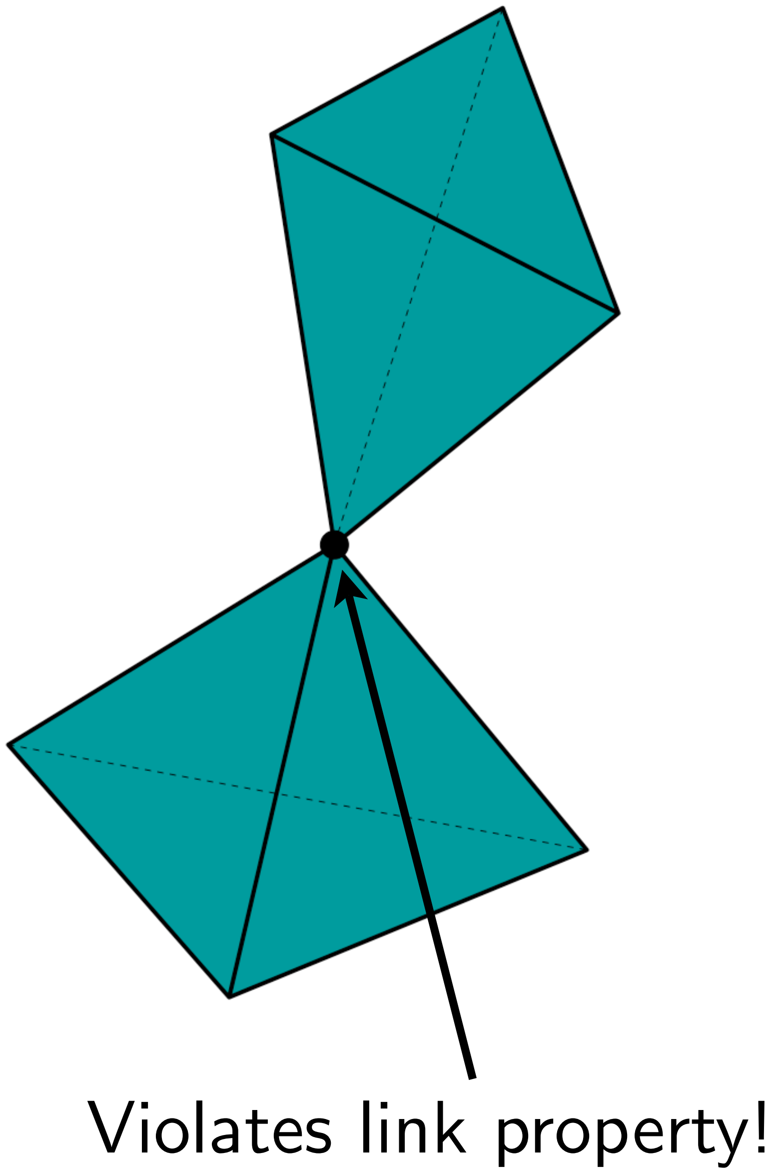

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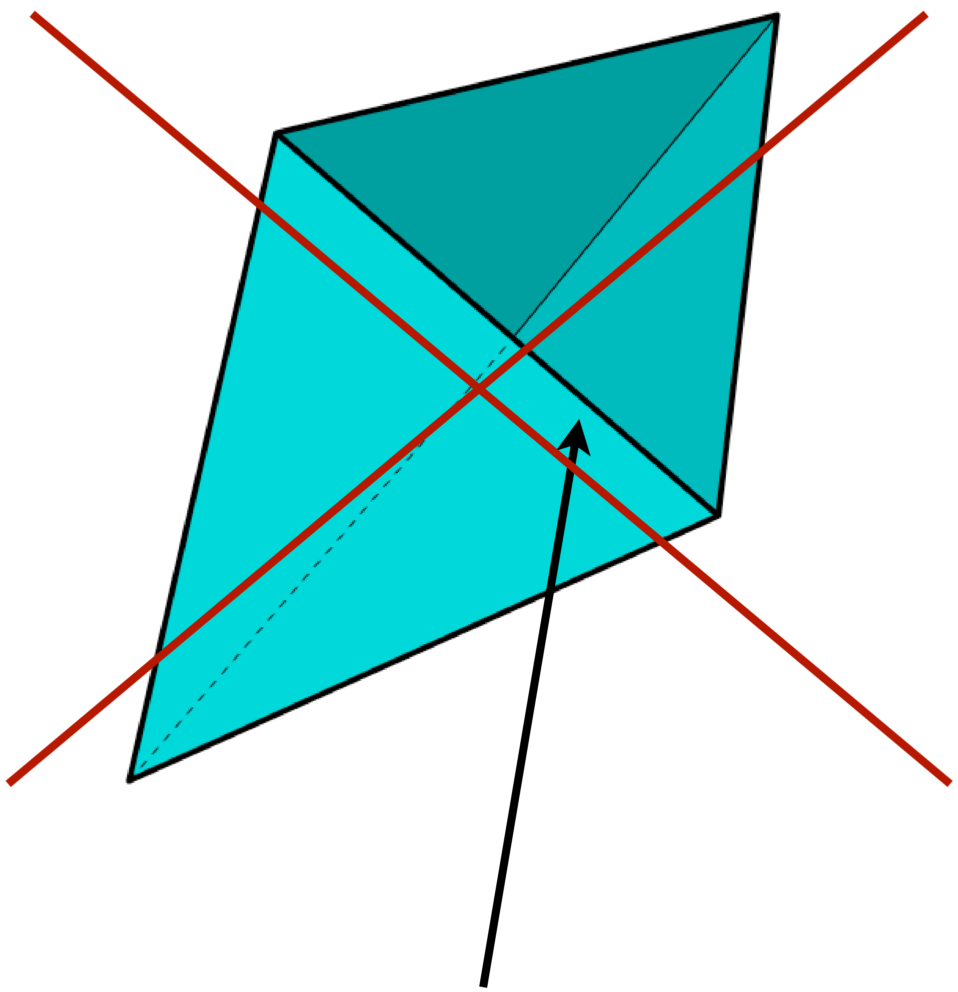

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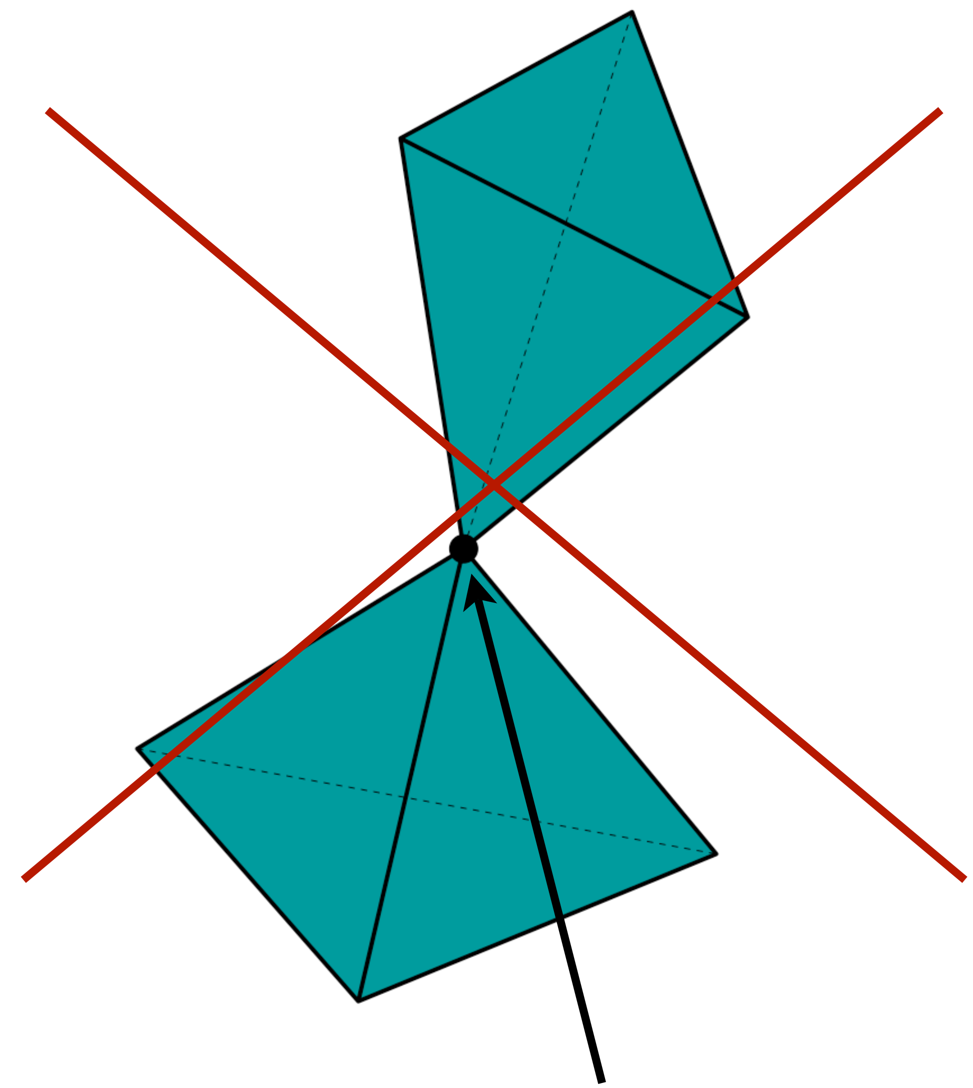

Violates link property!

They are NOT piecewise-linear surfaces

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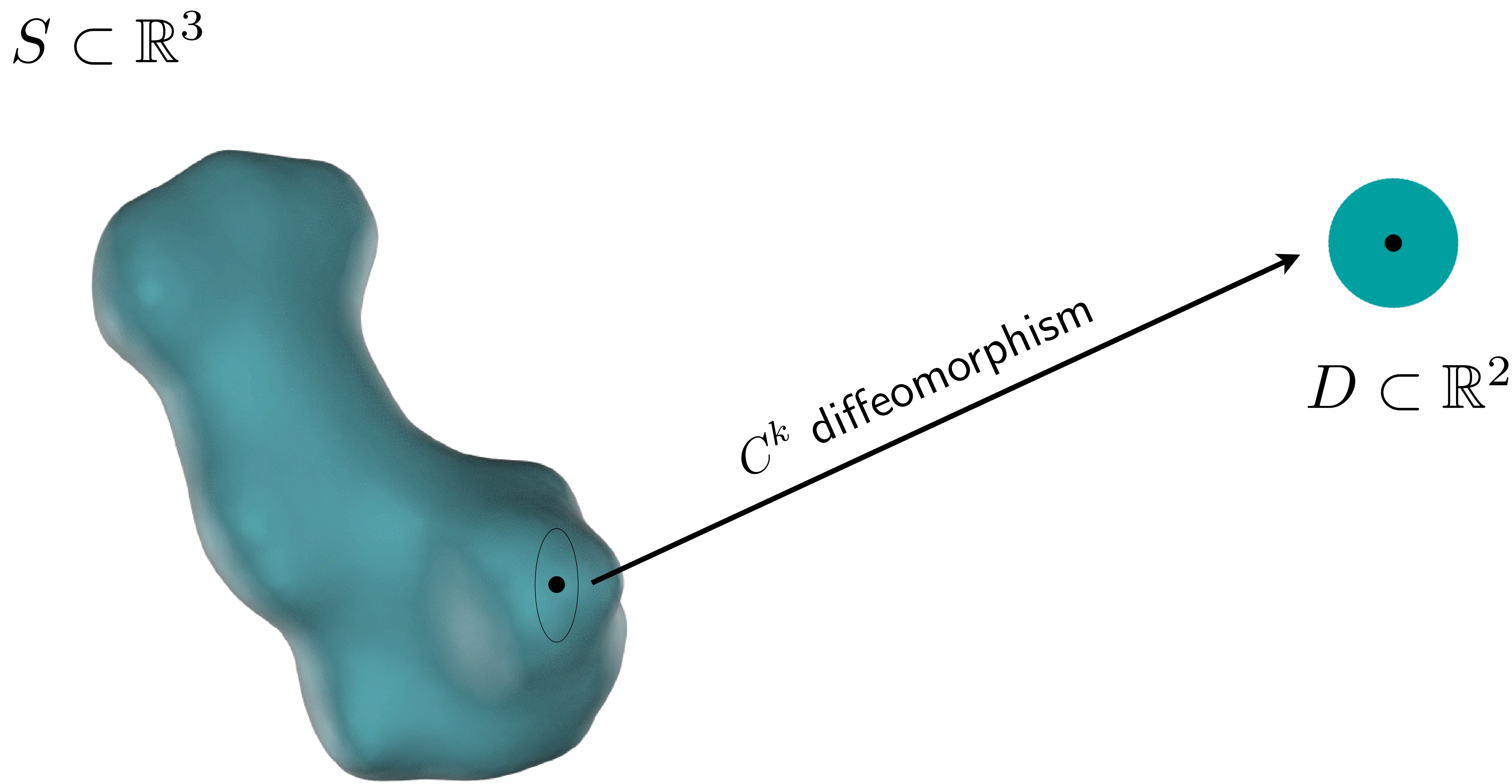

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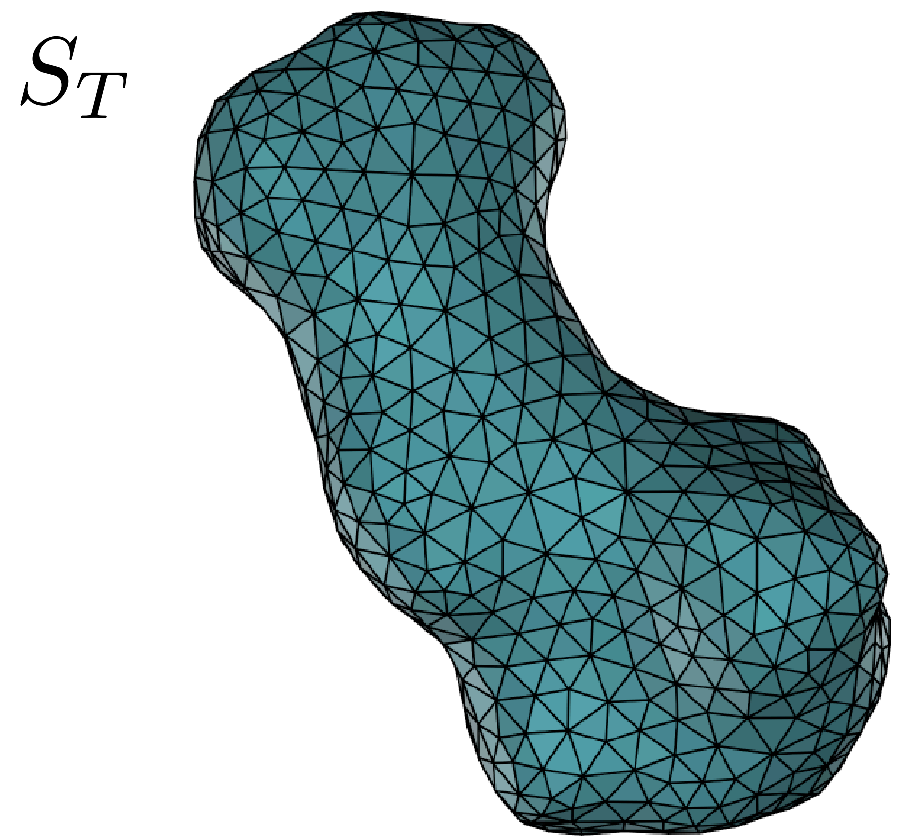

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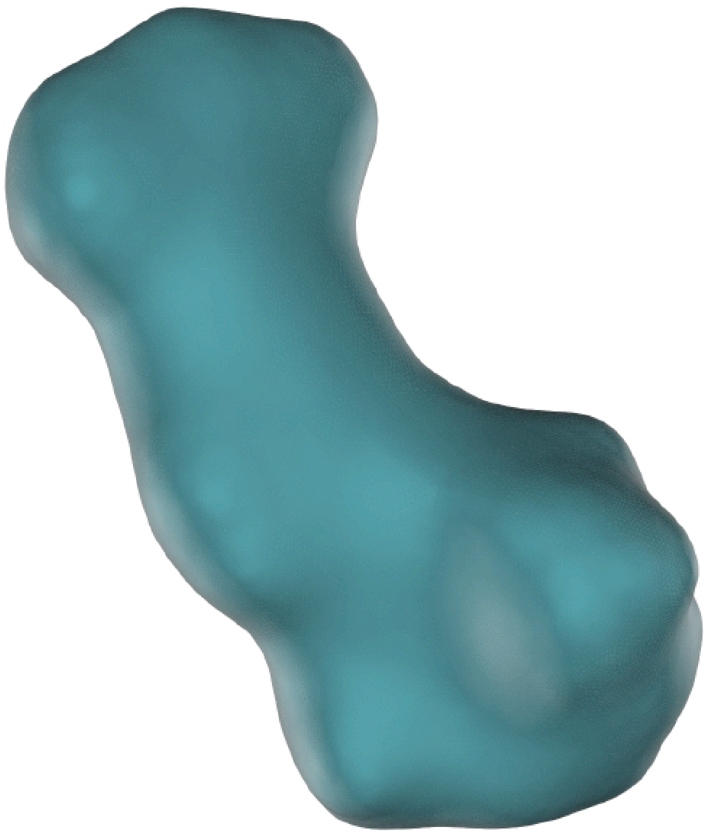

Topological and geometric guarantees!

The Surface Fitting Problem

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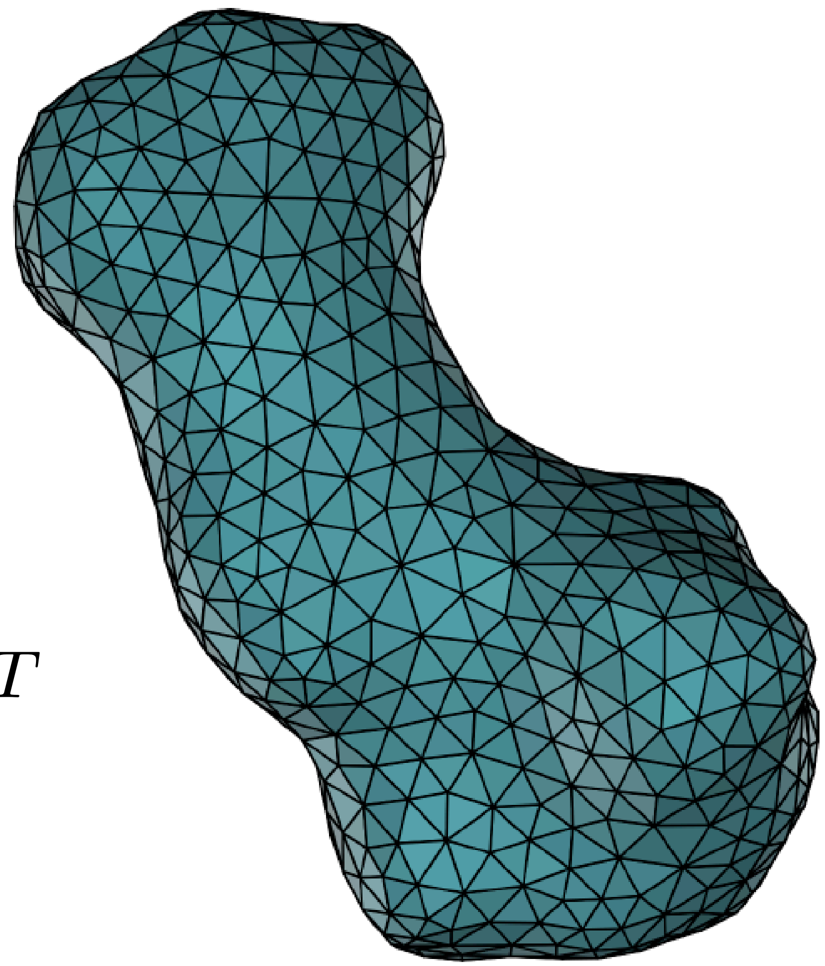
From now on, we will refer to S_T as a **polygonal mesh**.



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- Reasonably well-solved for $k = 1, 2$, **but not higher**.
- Higher values of k are desirable in many applications.

Traditional Approaches

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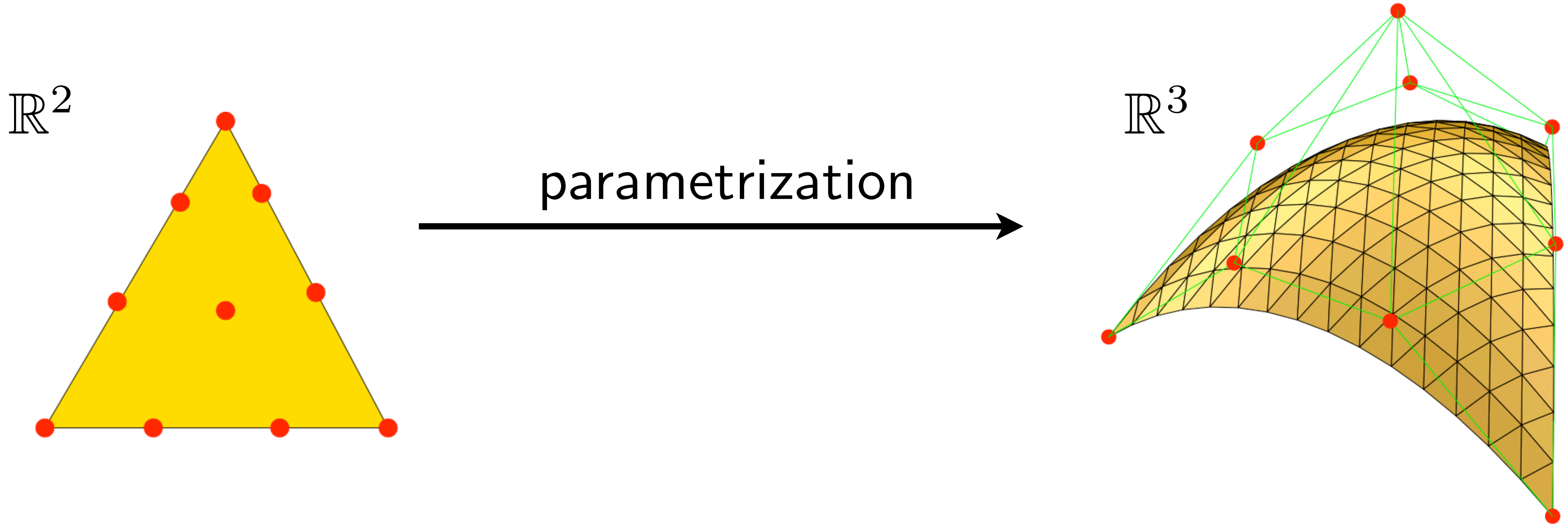

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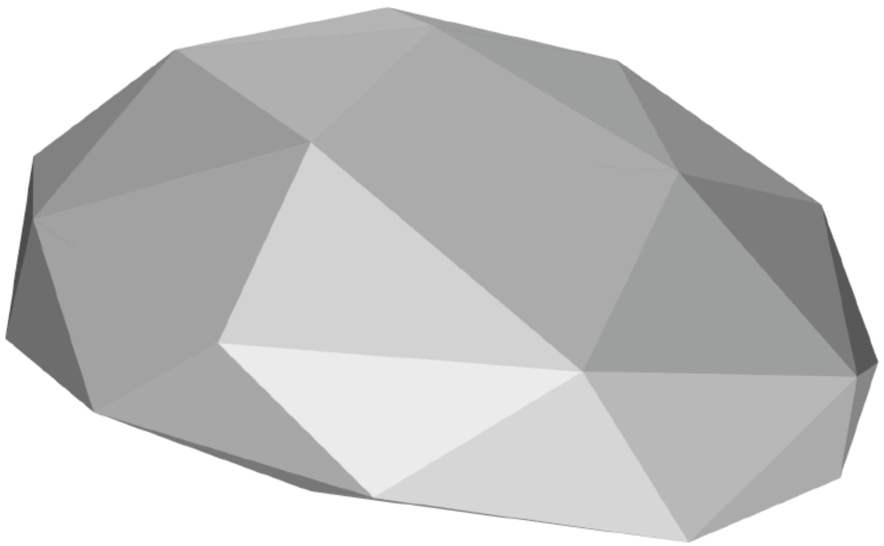

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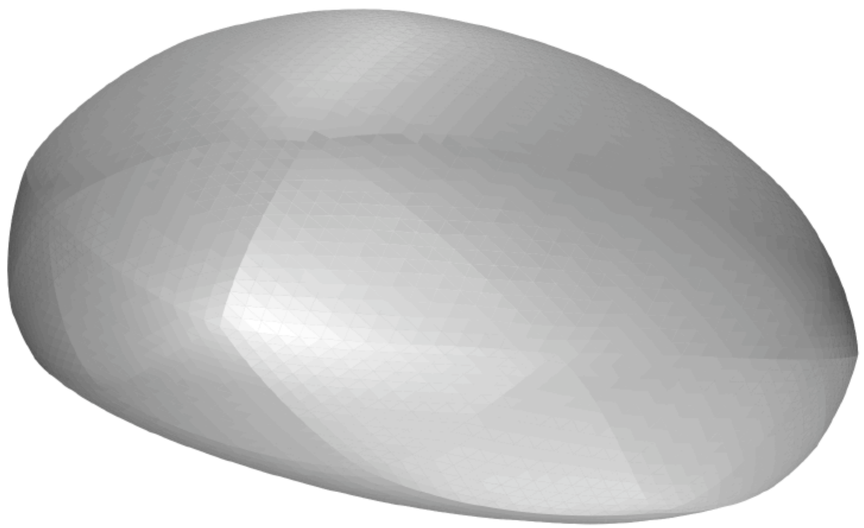

S

Continuity is enforced by control point placement!

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- Local control of geometry is not very flexible.

[Loop and DeRose, 1989], [Seidel, 1994], [Prautzsch, 1997], and [Reif, 1998] give C^k parametric approaches for arbitrary k.

Traditional Approaches

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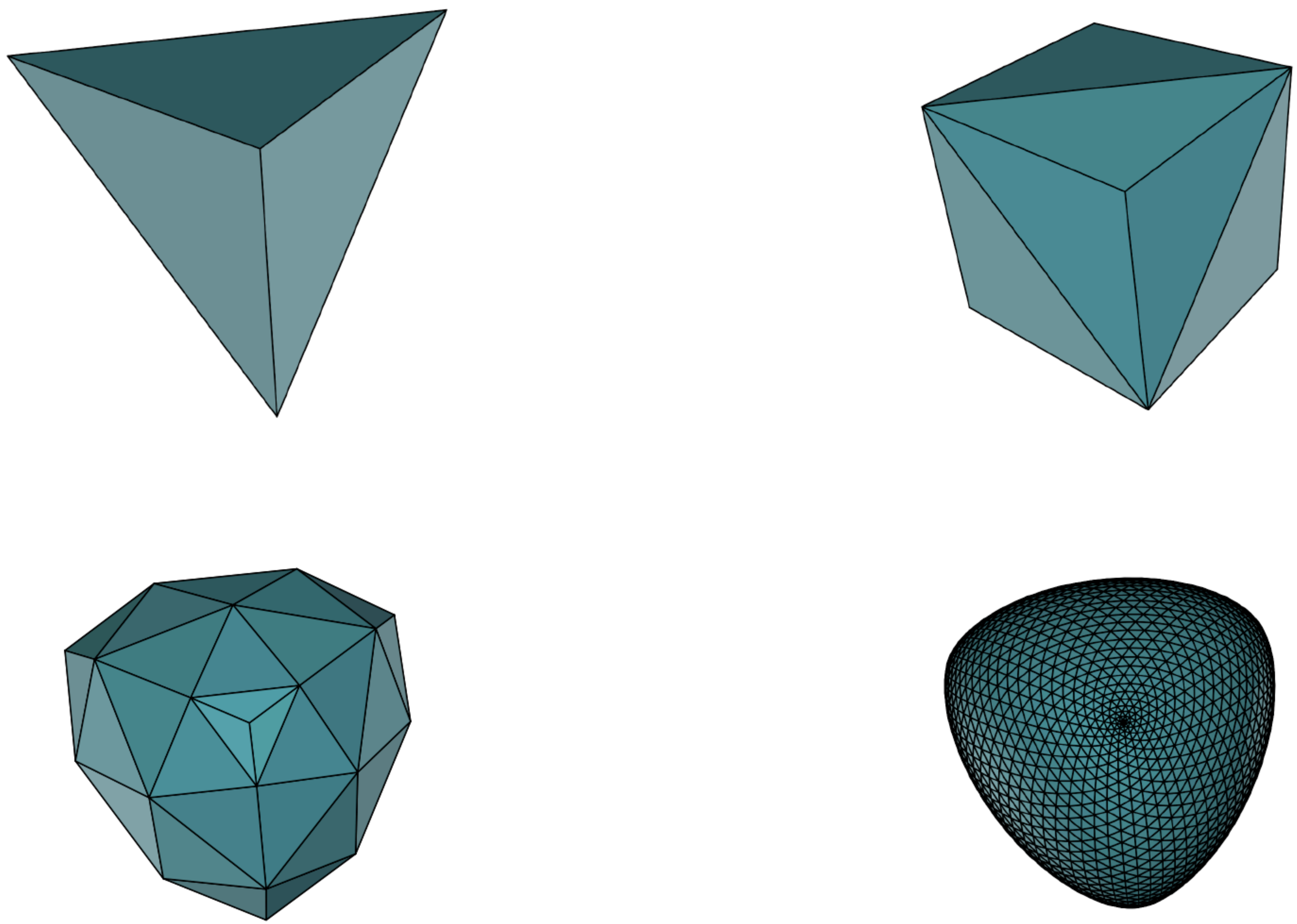
Another popular approach consists of using subdivision surfaces.



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See [Warren, 2002].

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Then, one can use RBF, MPU, moving least squares, etc.

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See [Shen, O'Brien, and Shewchuk, 2004] and [Kolluri, 2005].

Implicit and parametric surfaces have complementary features.

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The manifold approach has also some advantages over the traditional approaches when it comes to certain applications, such as texture synthesis and the solution of equations on surfaces.

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- describe the manifold-based approach for the surface fitting problem,

- review the main existing solutions and their limitations, and

- point out some applications and research challenges in Computer Graphics, Image Processing, and Computer Vision that can be more naturally tackled by using manifolds.

What's Next?

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II. Manifolds

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III. Constructing Manifolds

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IV. Fitting Surfaces to Polygonal Meshes – Part I

What's Next?

II. Manifolds

III. Constructing Manifolds

IV. Fitting Surfaces to Polygonal Meshes – Part I

Coffee break

What's Next?

What's Next?

V. Fitting Surfaces to Polygonal Meshes – Part II

What's Next?

V. Fitting Surfaces to Polygonal Meshes – Part II

VI. Adaptive Manifold Fitting – Part I

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V. Adaptive Manifold Fitting – Part II

VIII. Applications of Manifolds and Research Challenges

Manifolds

Jean Gallier
UPenn

Outline

- Manifolds: Brief History
- Informal definition
- Formal definition
- Examples
  - The Sphere
  - Real Projective Space
- Conclusions

Origins of Manifolds

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- In a famous paper published in 1888, Dyck already uses the term **manifold** (in German).
- In the early 1900's, Dehn, Heegaard, Veblen and Alexander routinely used the term **manifold**.
- Hermann Weyl was among the first to give a rigorous definition (1913).

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Hermann Klaus Hugo Weyl
1885-1955

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Hassler Whitney
1907-1989

Manifold: An Intuitive Picture

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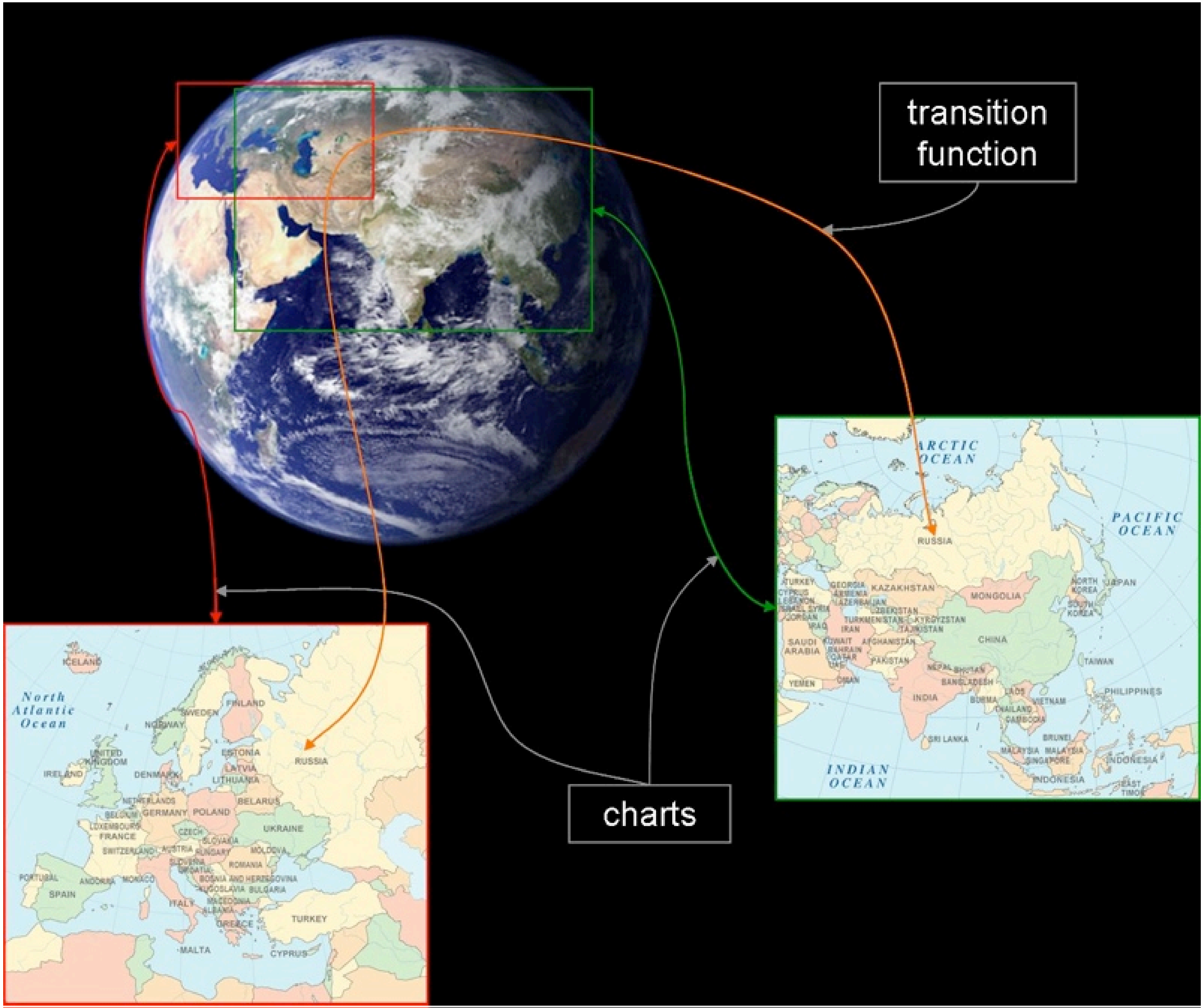

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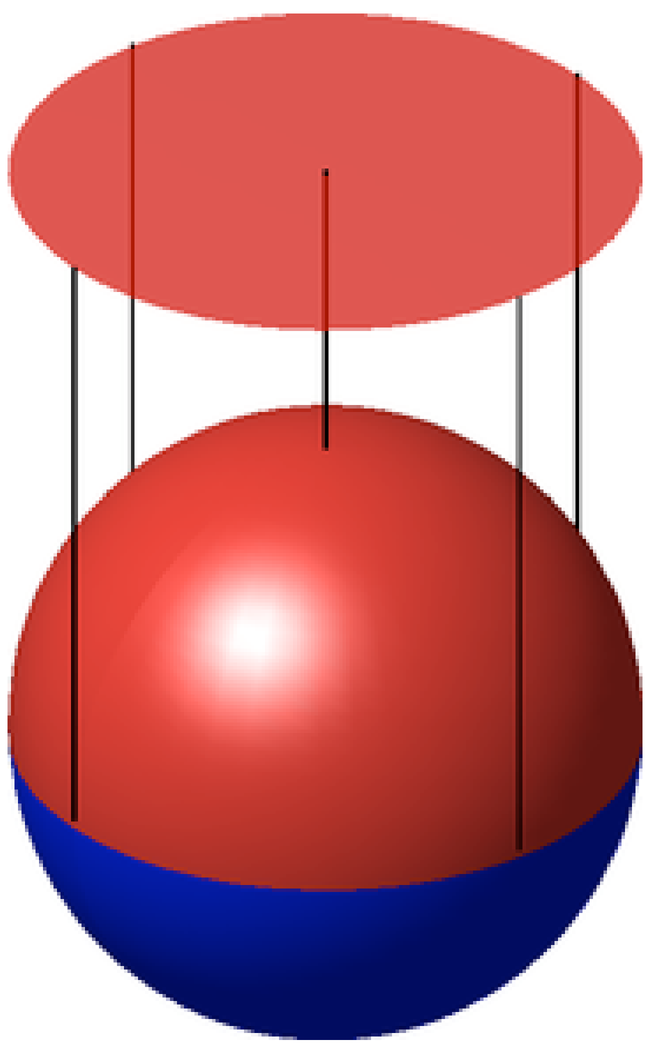

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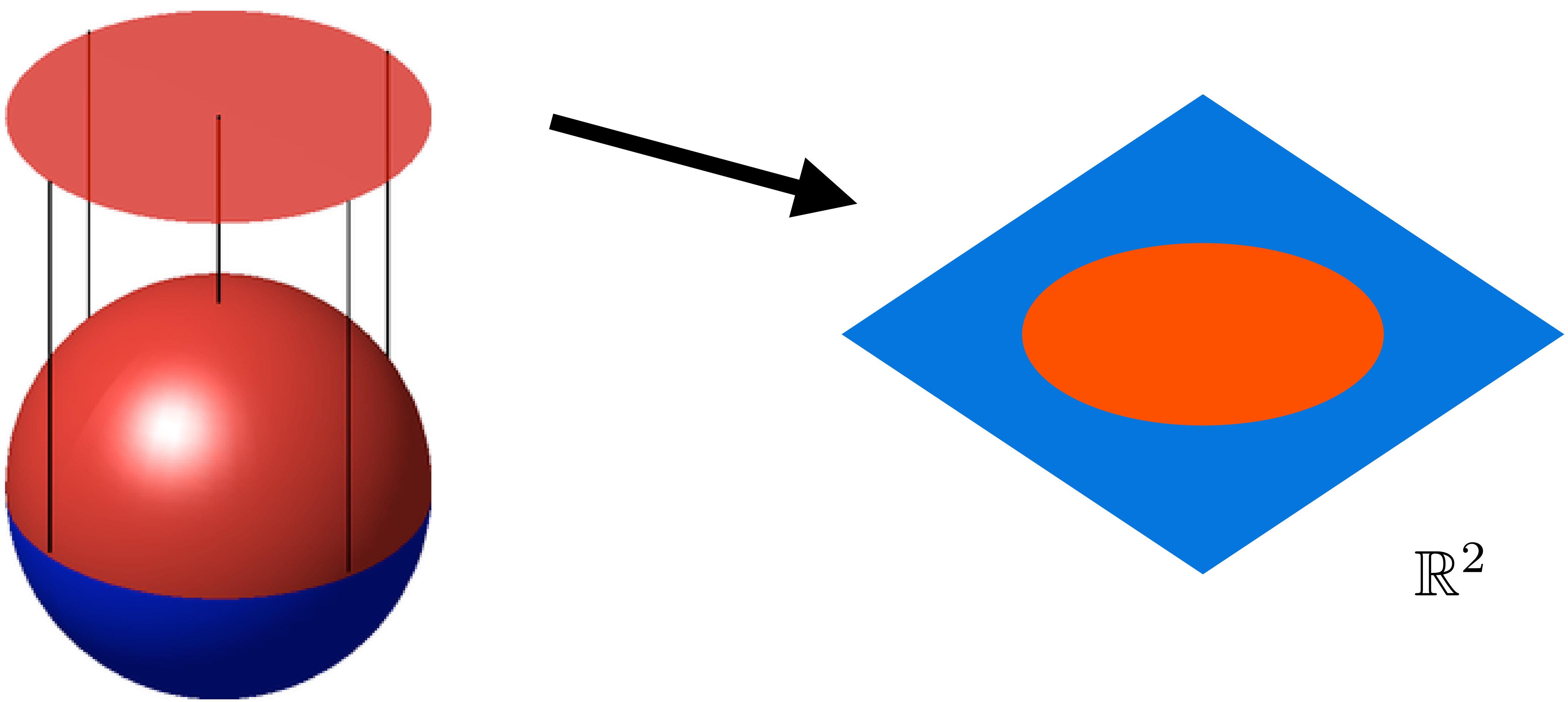

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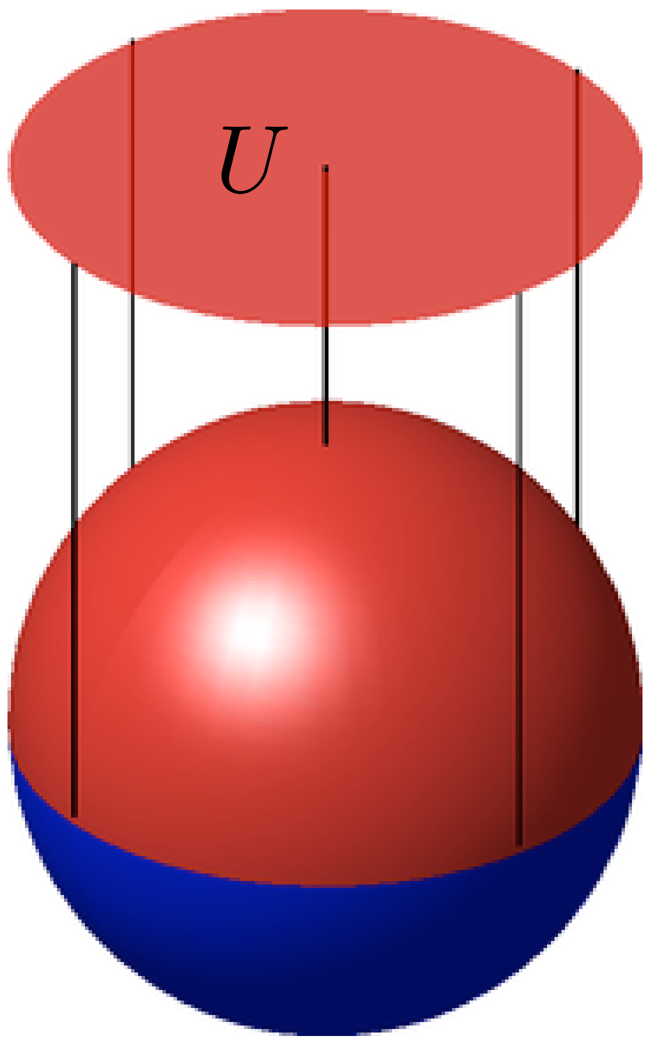

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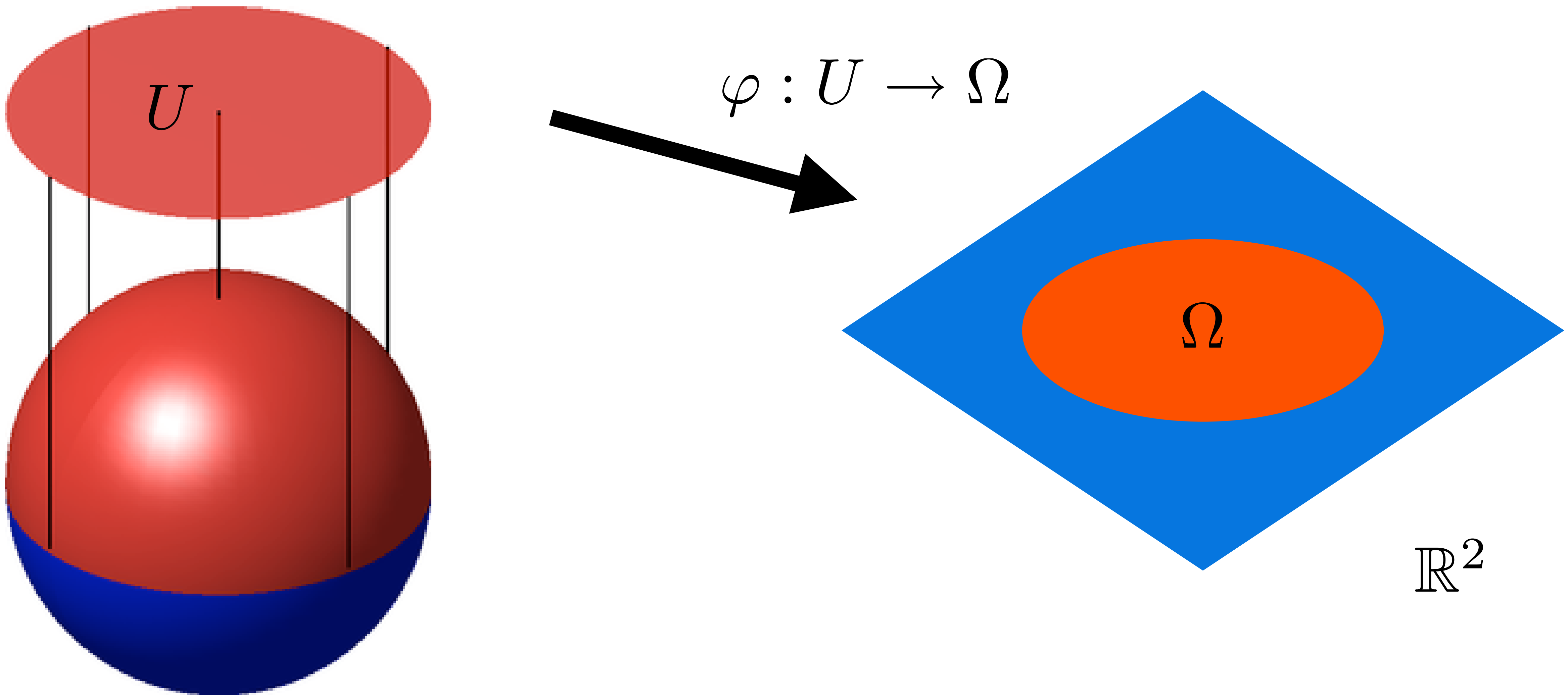

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- We also want to be able "to do calculus" on our manifolds. For this we need some conditions on **overlaps** of open sets.

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- Whenever $U_i \cap U_j \neq \emptyset$, we need some condition on the **transition function**,

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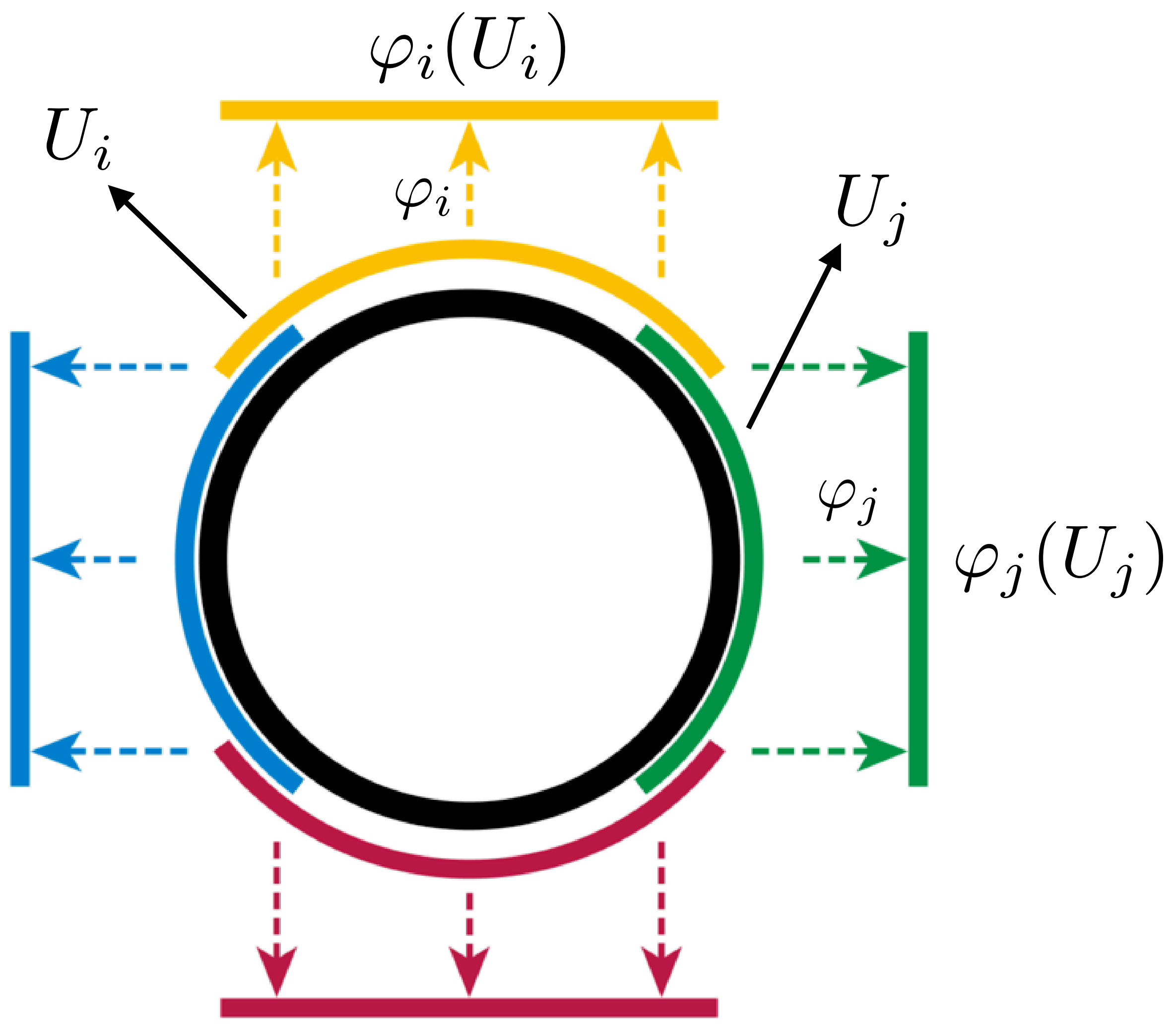

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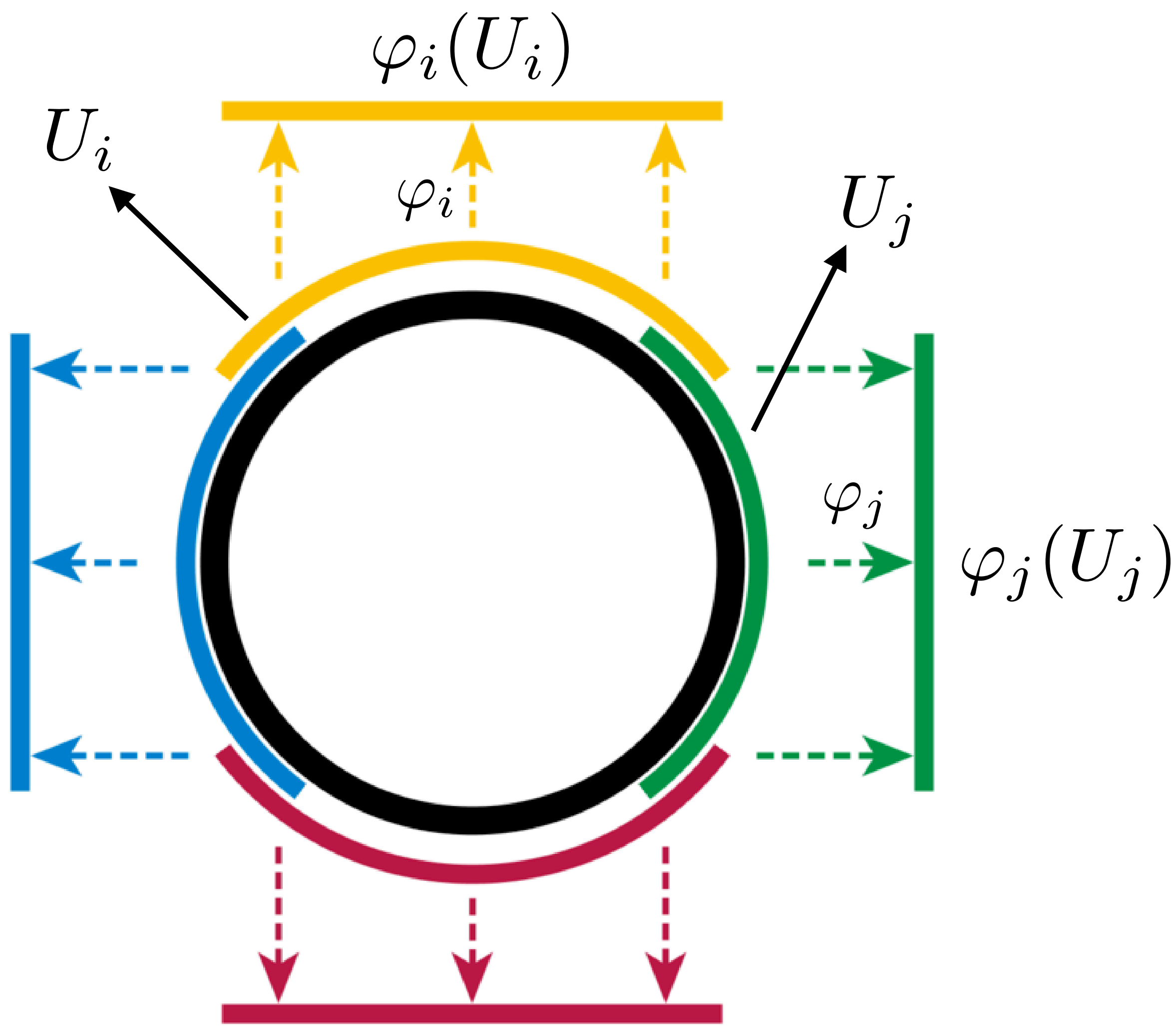

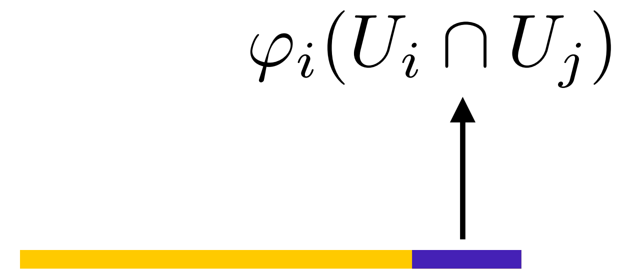

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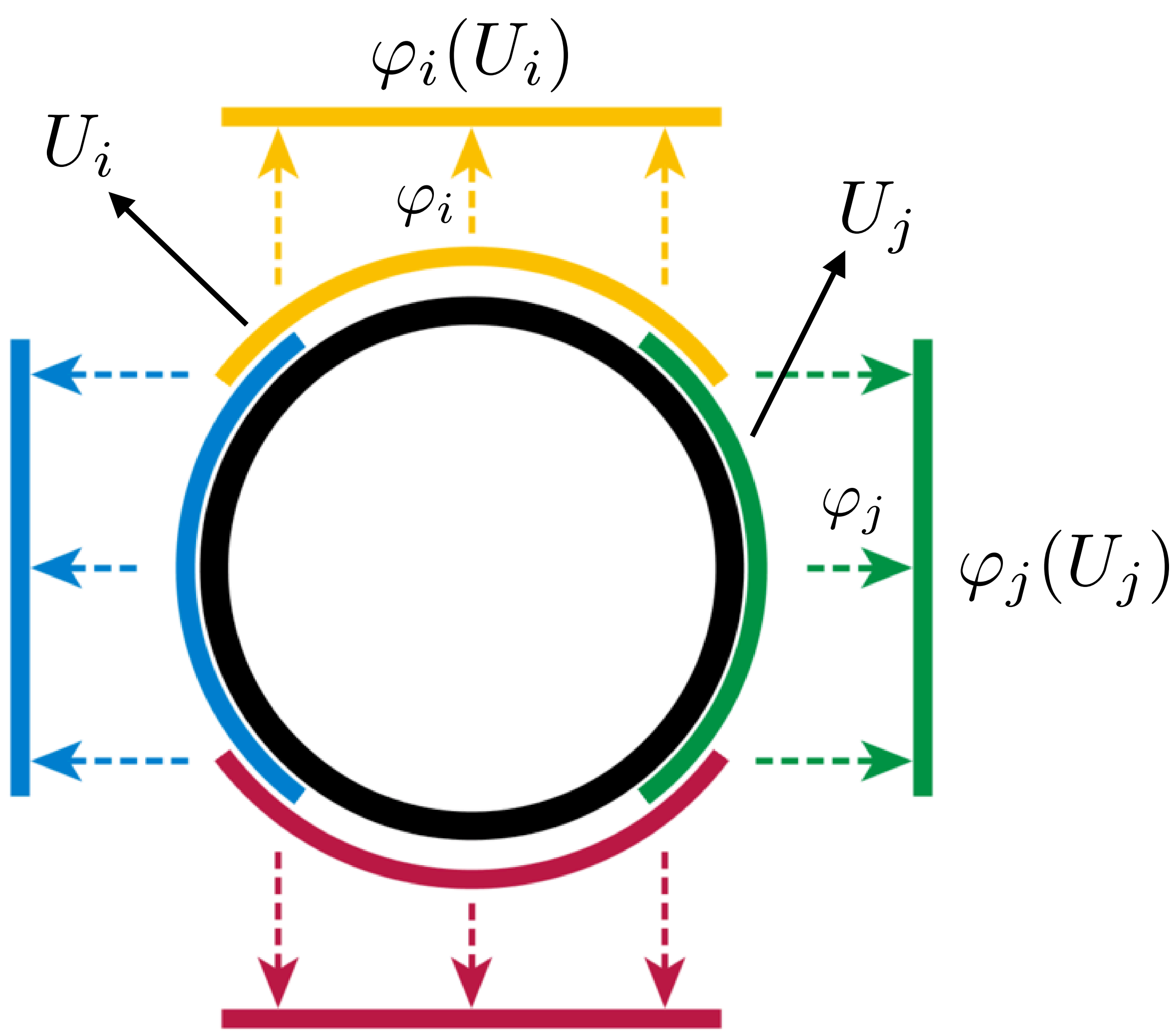

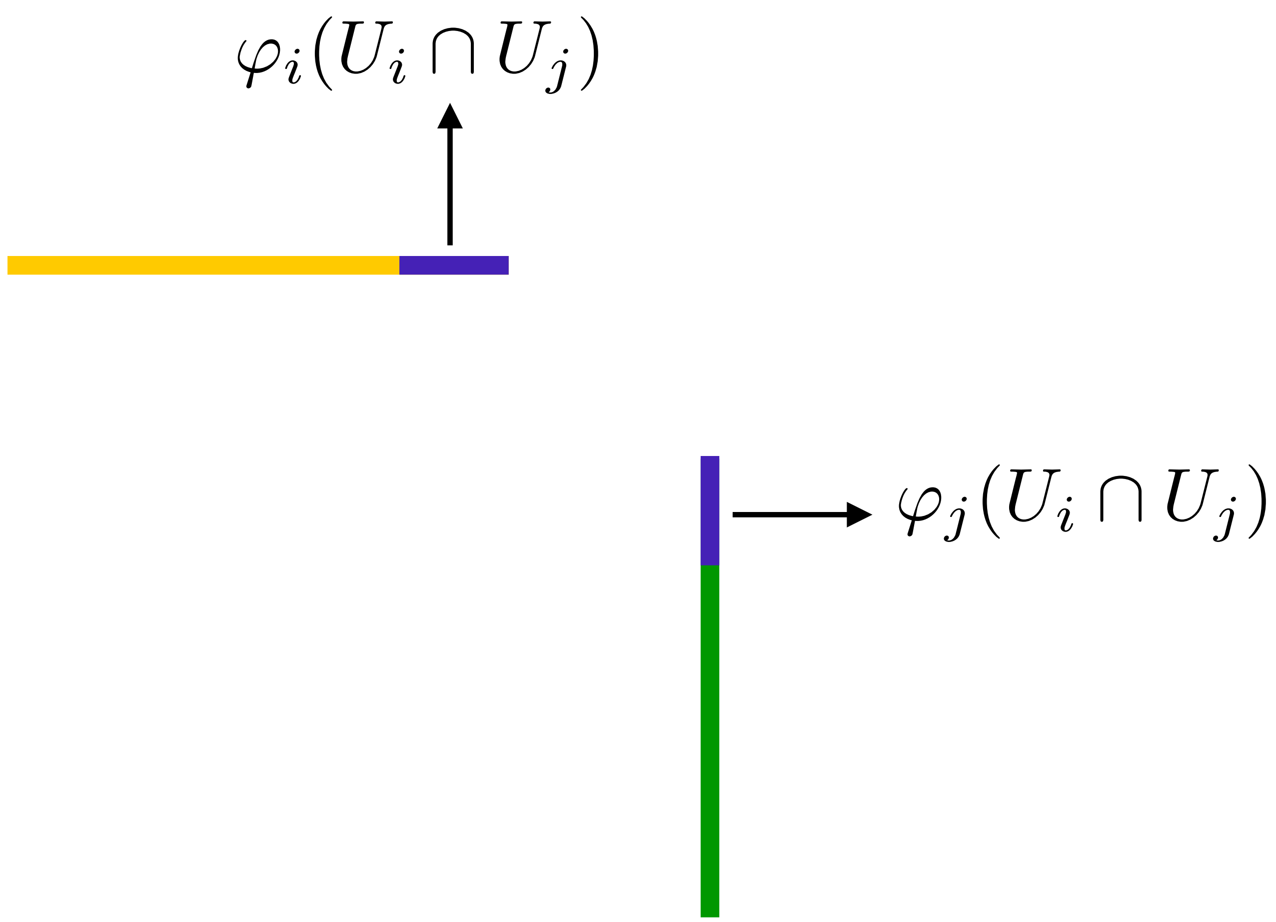

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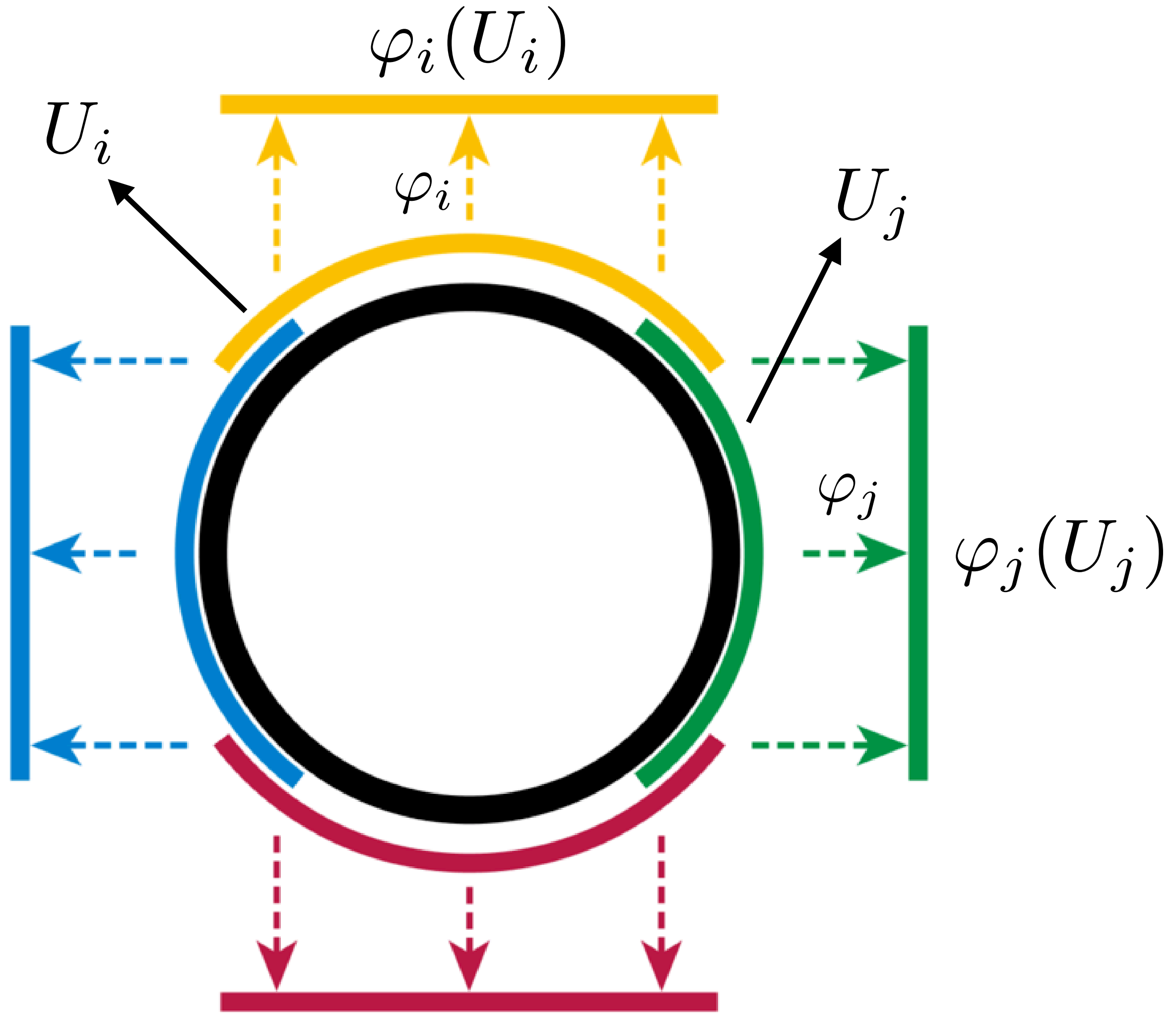

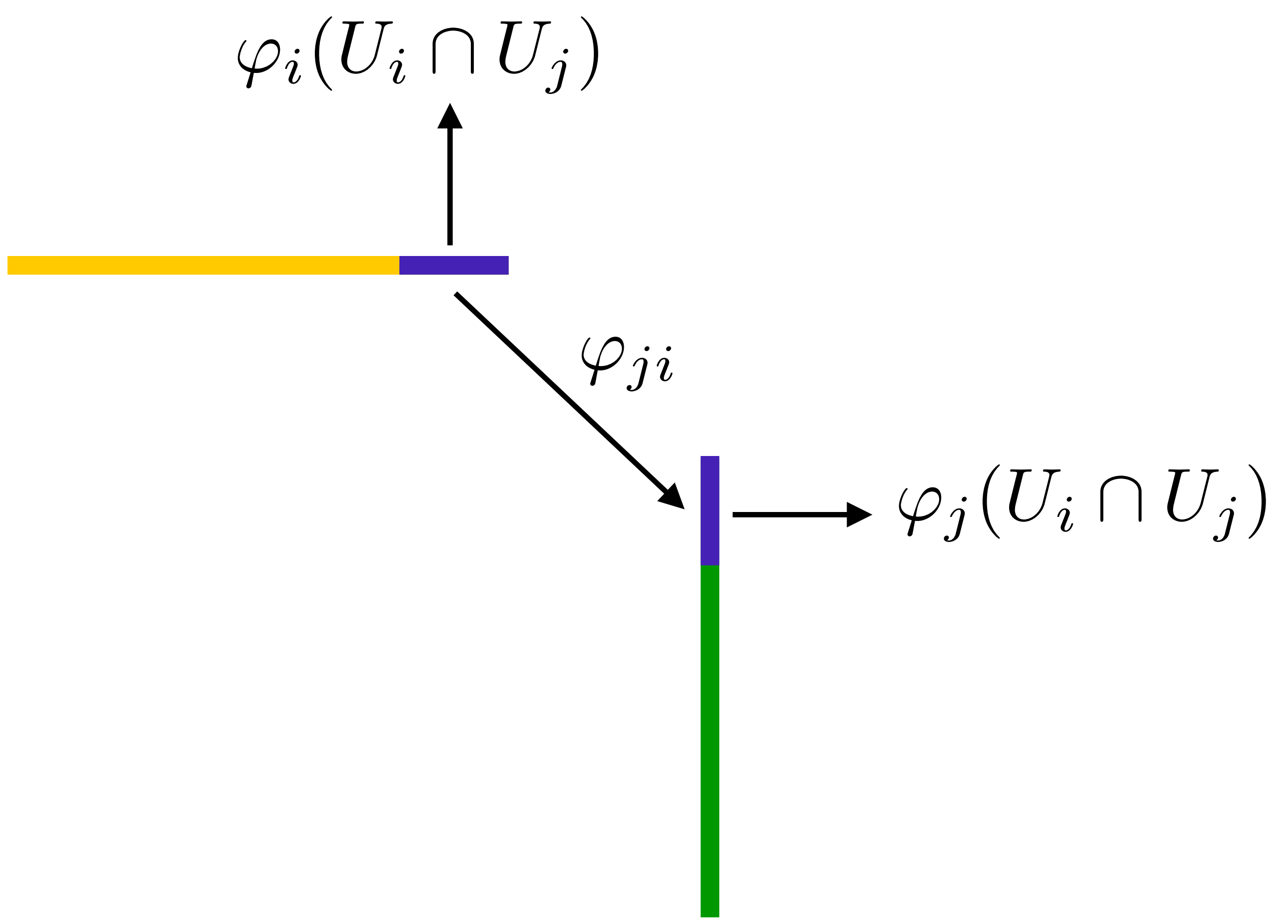

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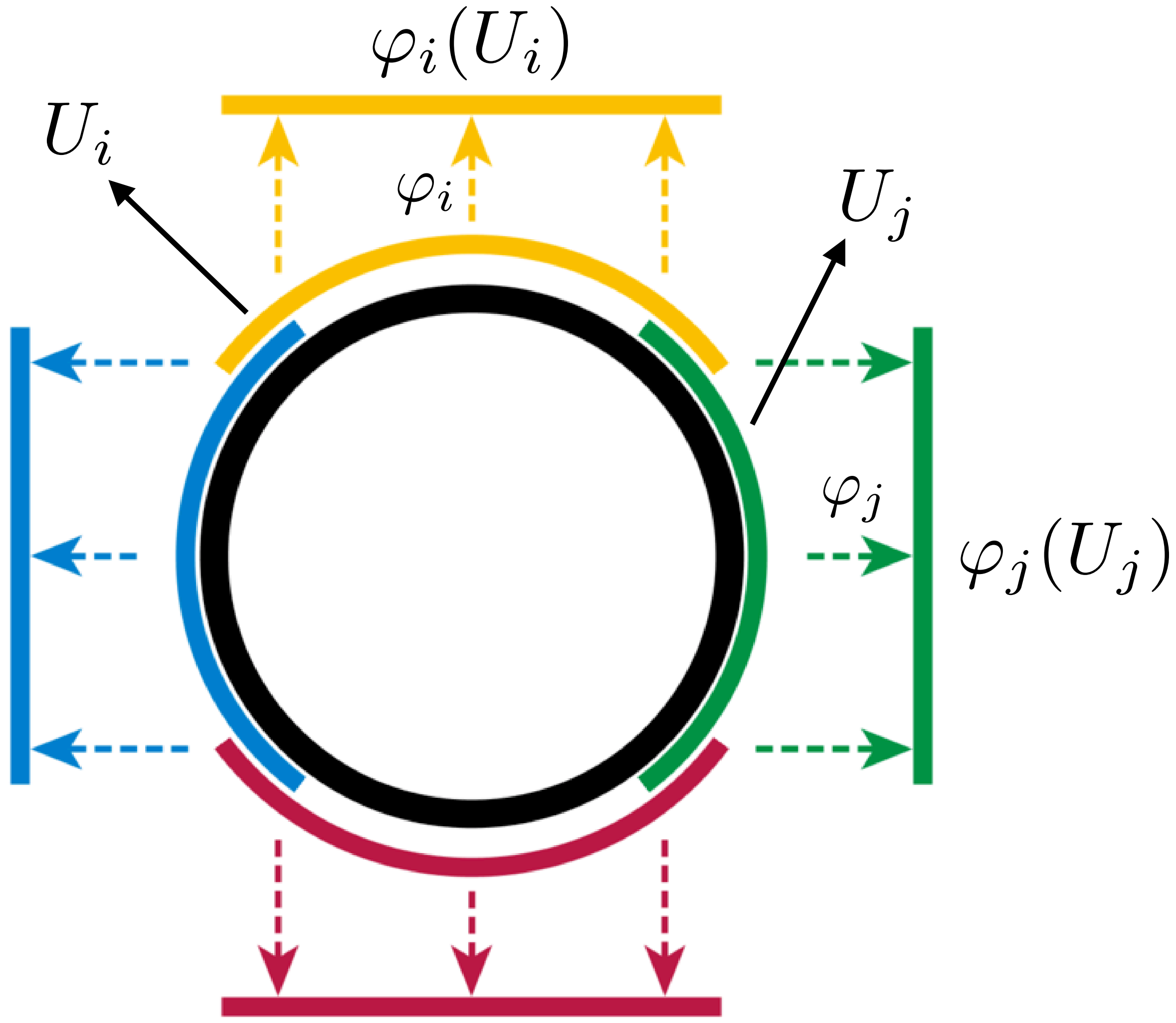
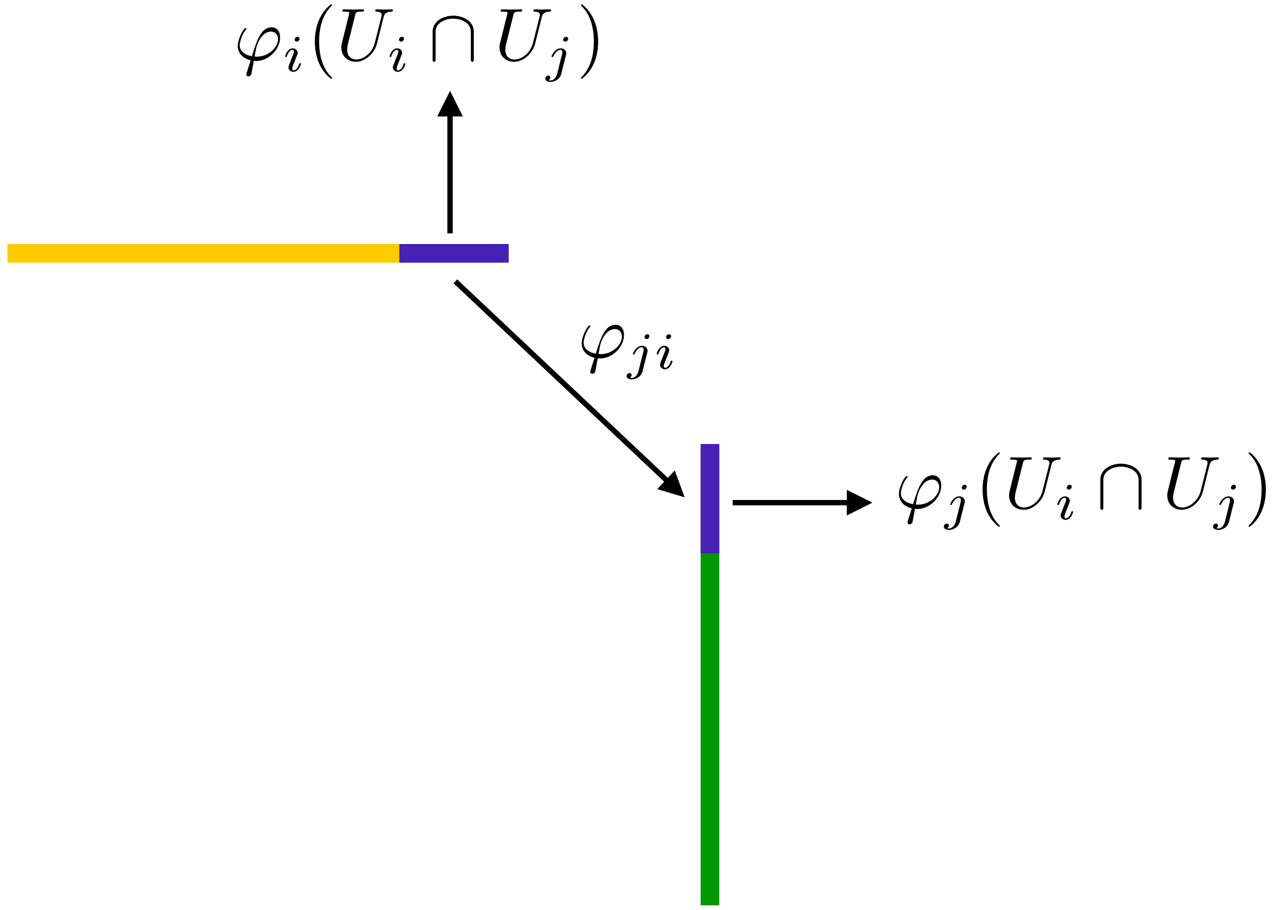
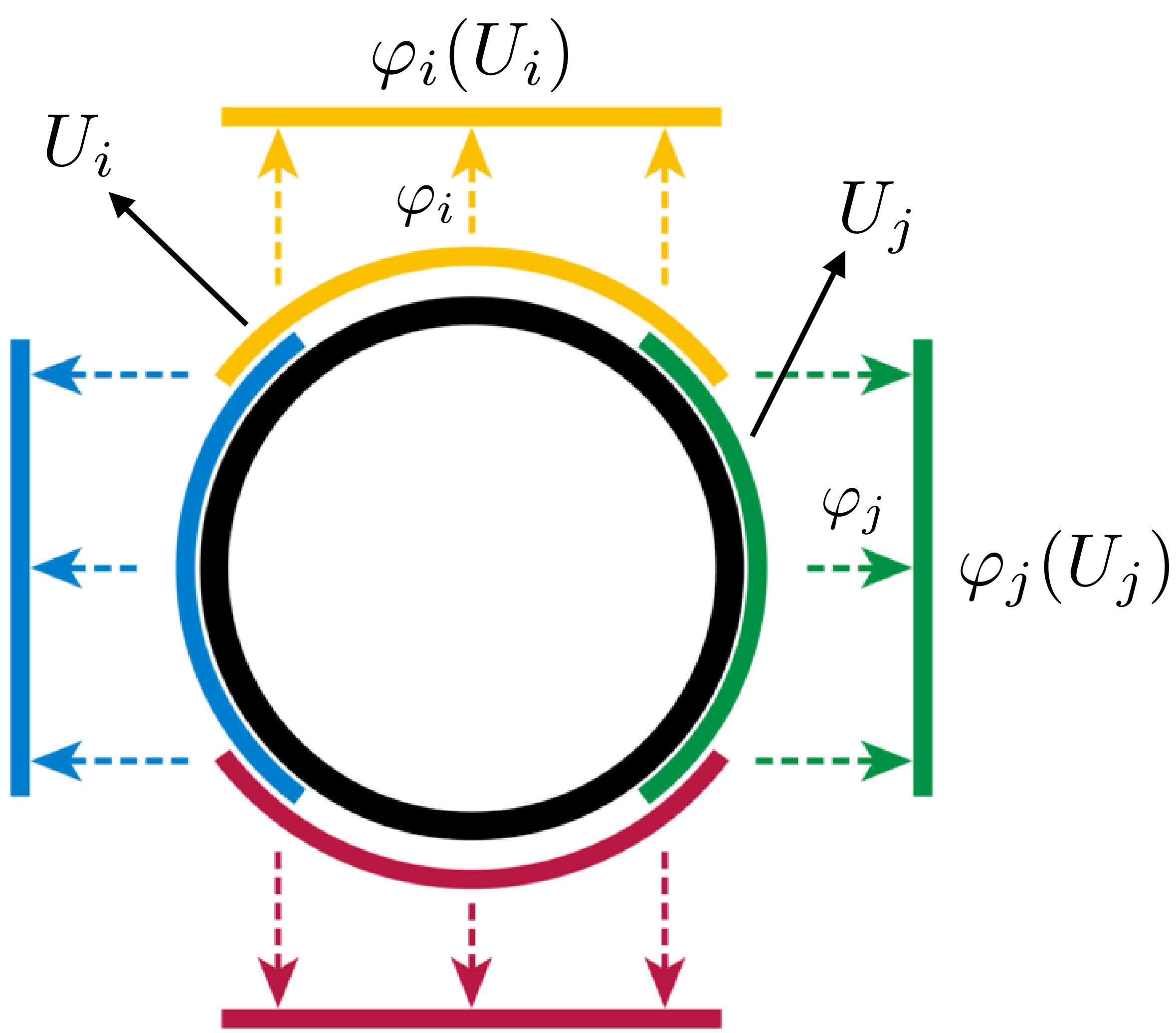
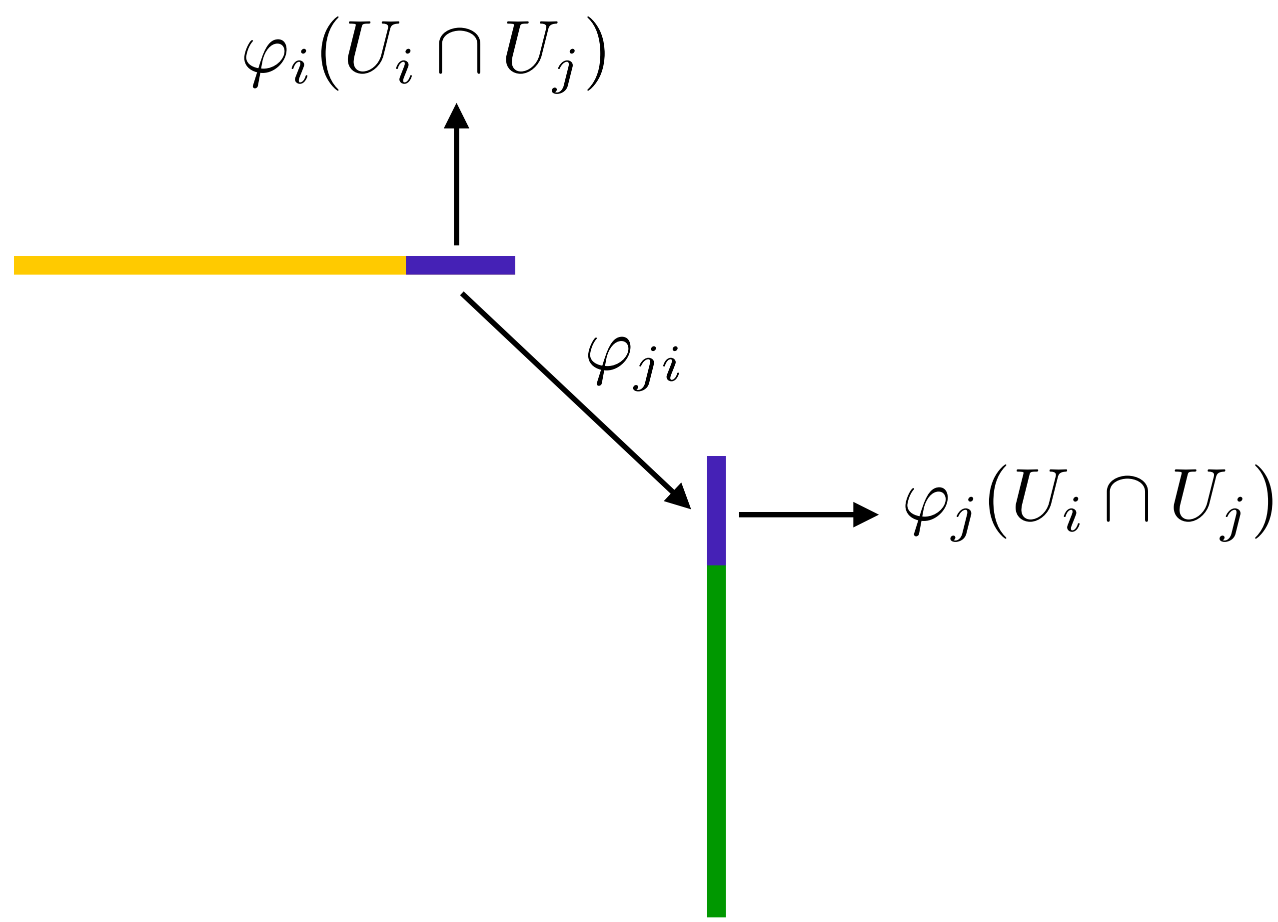




Manifolds: Informal Definition

- This is a map between two open subsets of $\mathbb{R}^n$ and we require it possess a certain amount of **smoothness**.





Manifolds: Formal Definition

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Recall the definition of a manifold...

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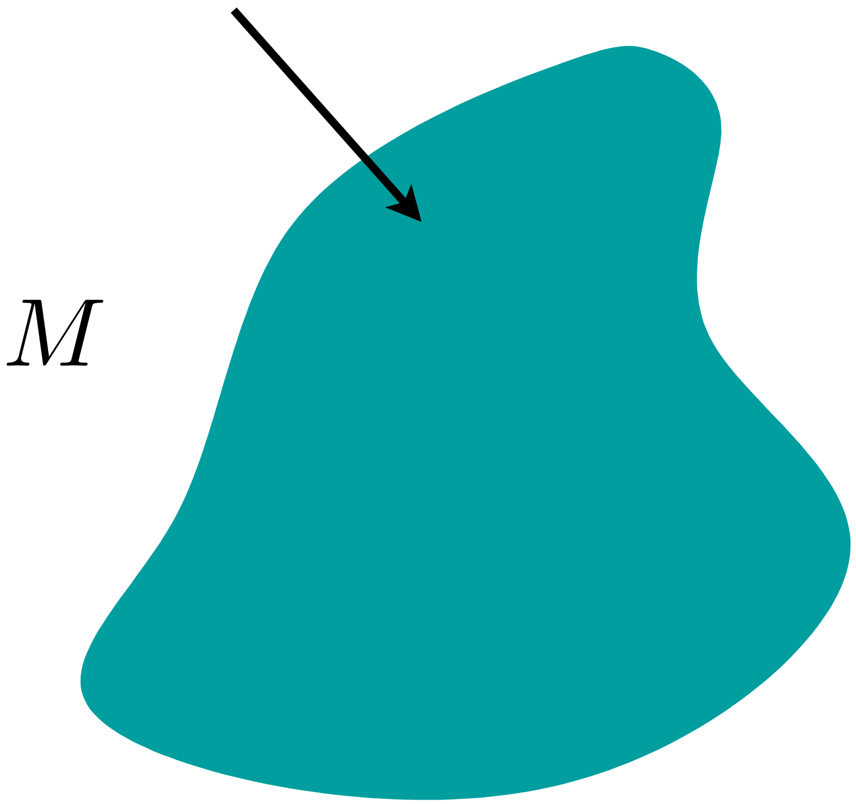
topological space



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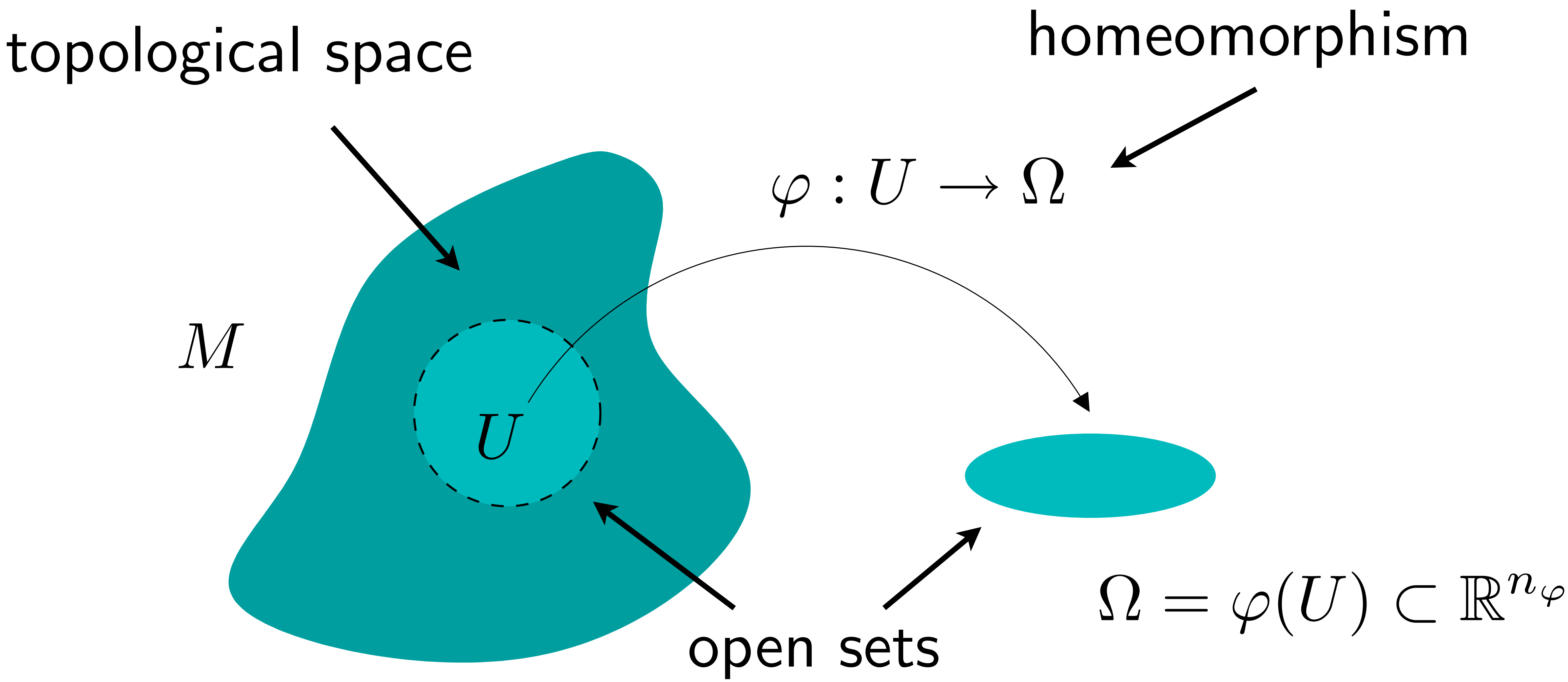

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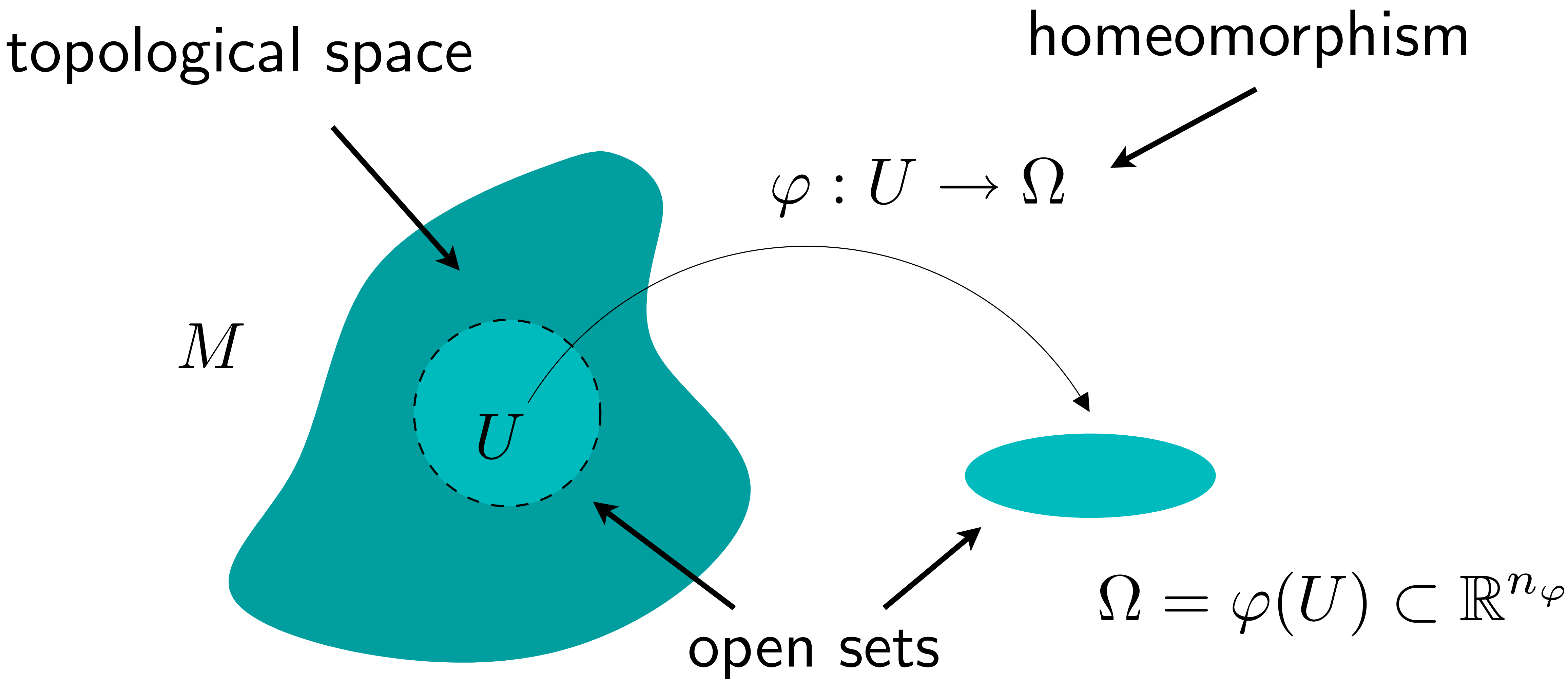

(U, φ) is called a **chart**.

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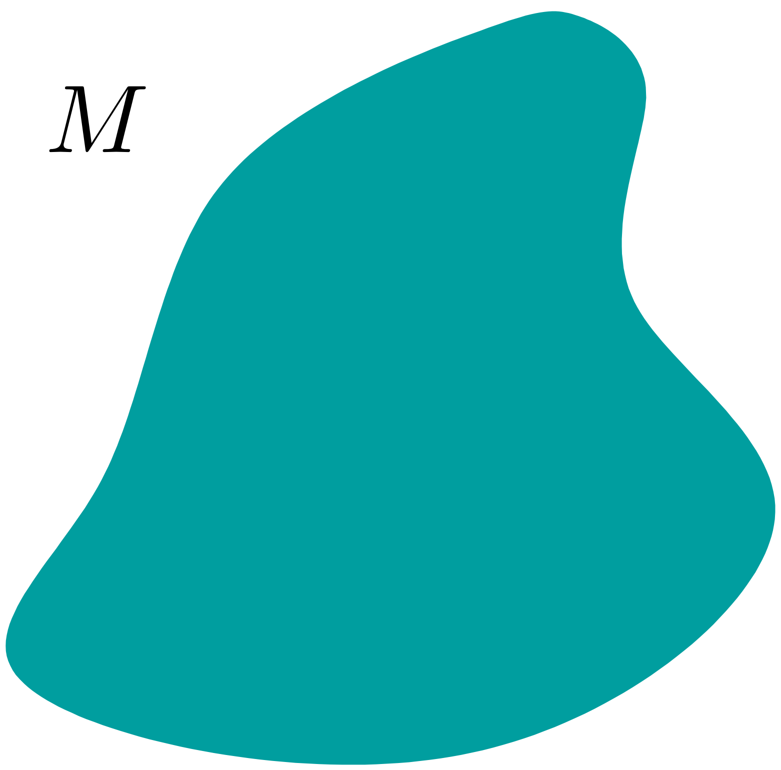

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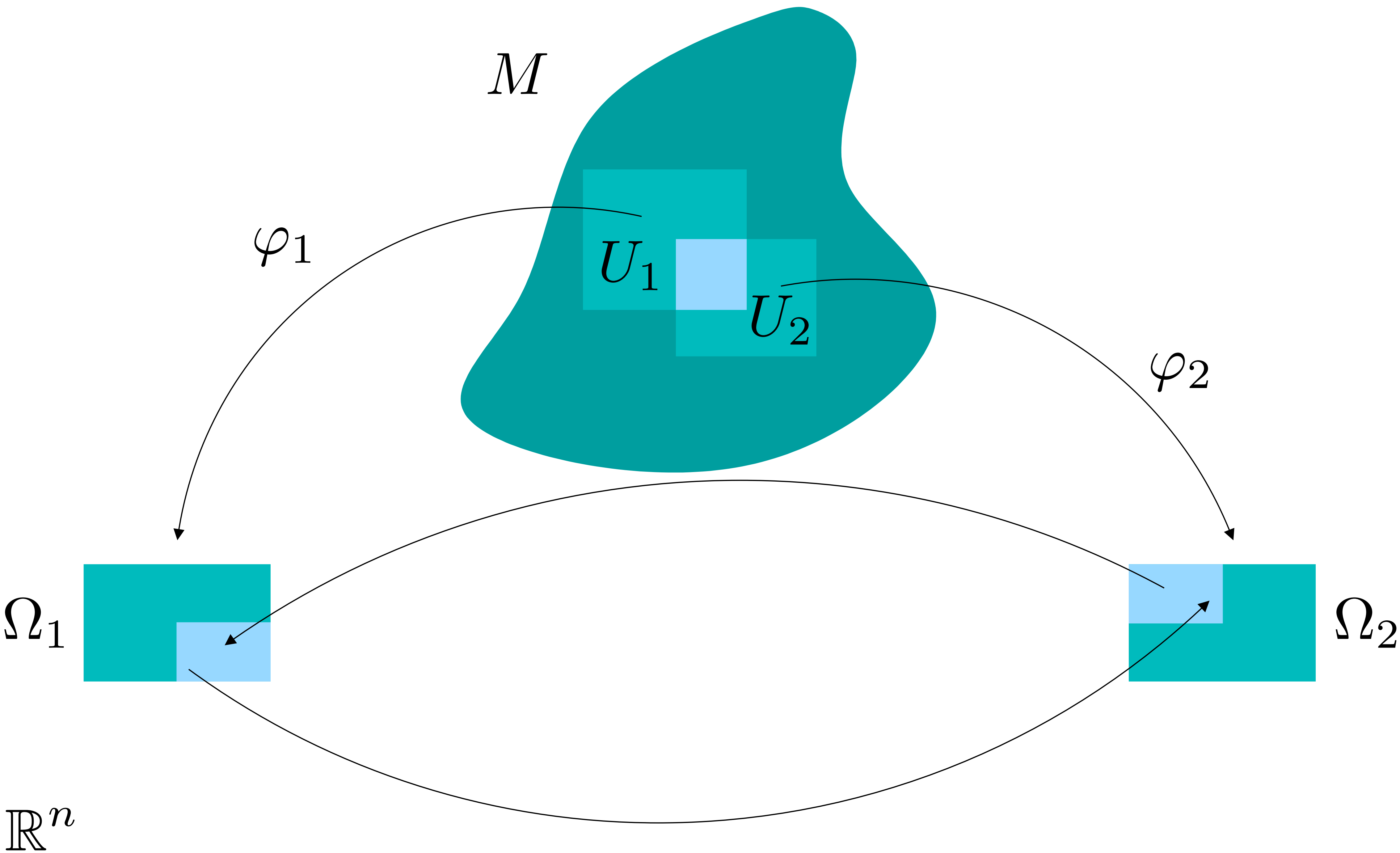

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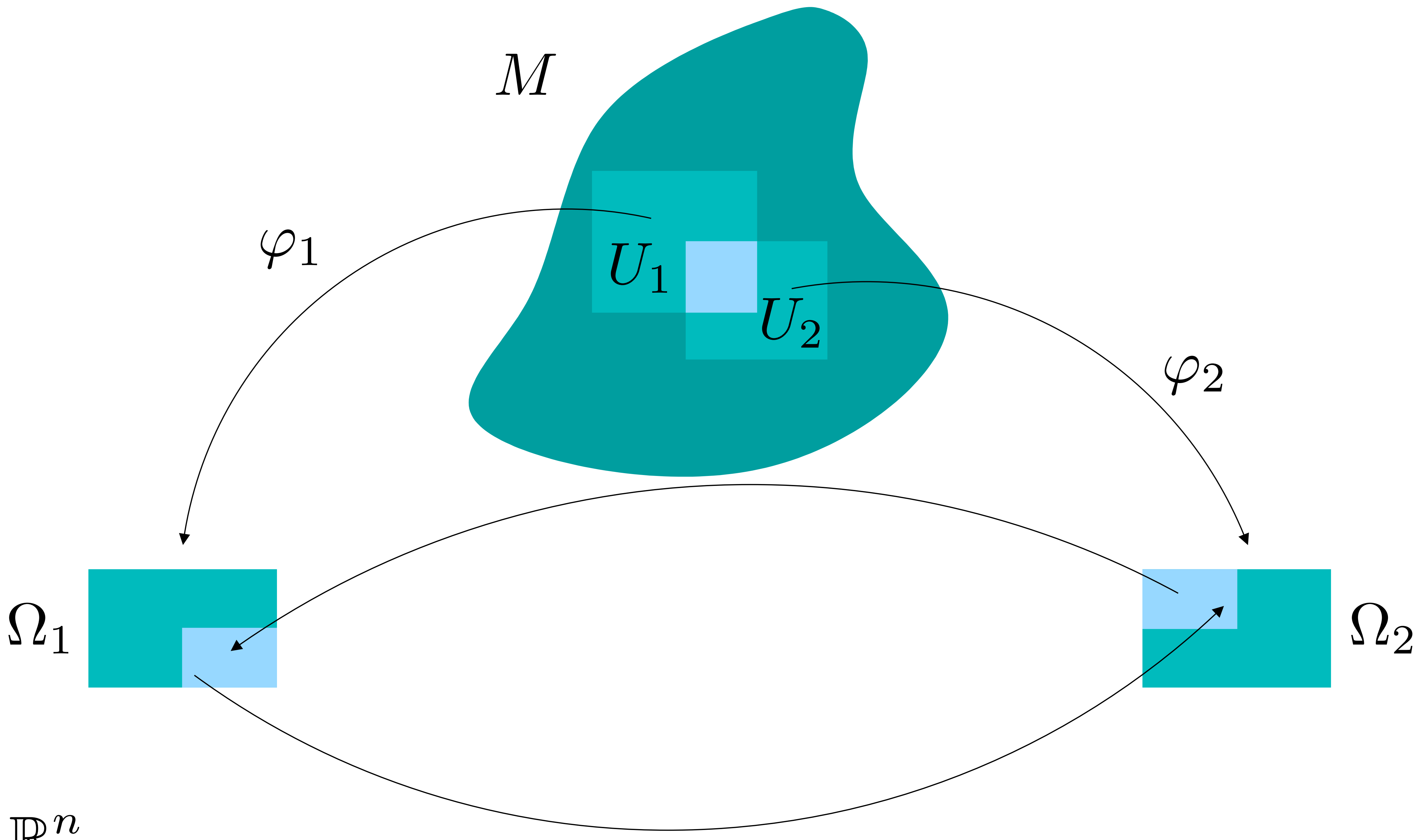

$$\varphi_{21} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$
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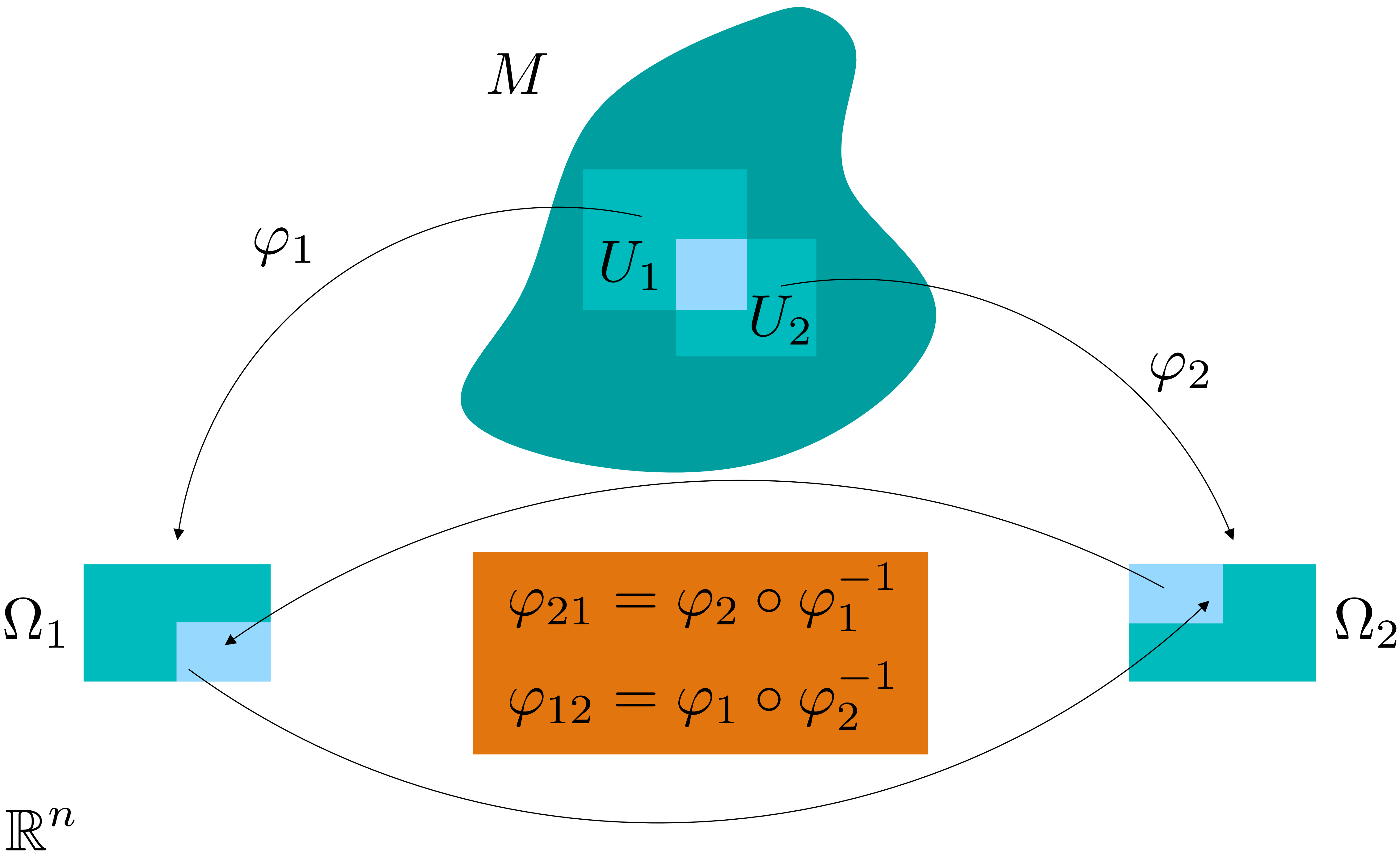

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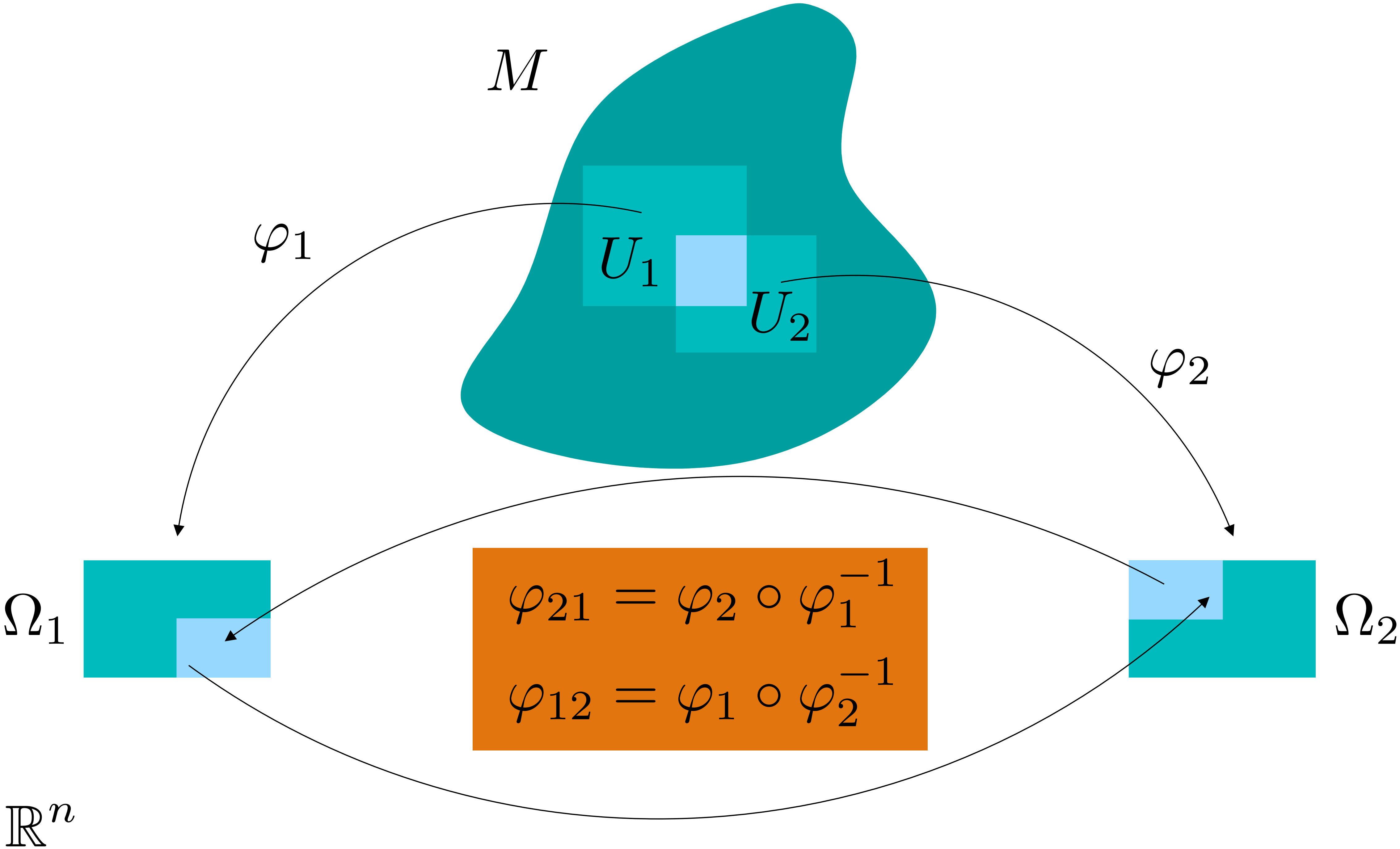

$$\varphi_{21} = \varphi_2 \circ \varphi_1^{-1}$$

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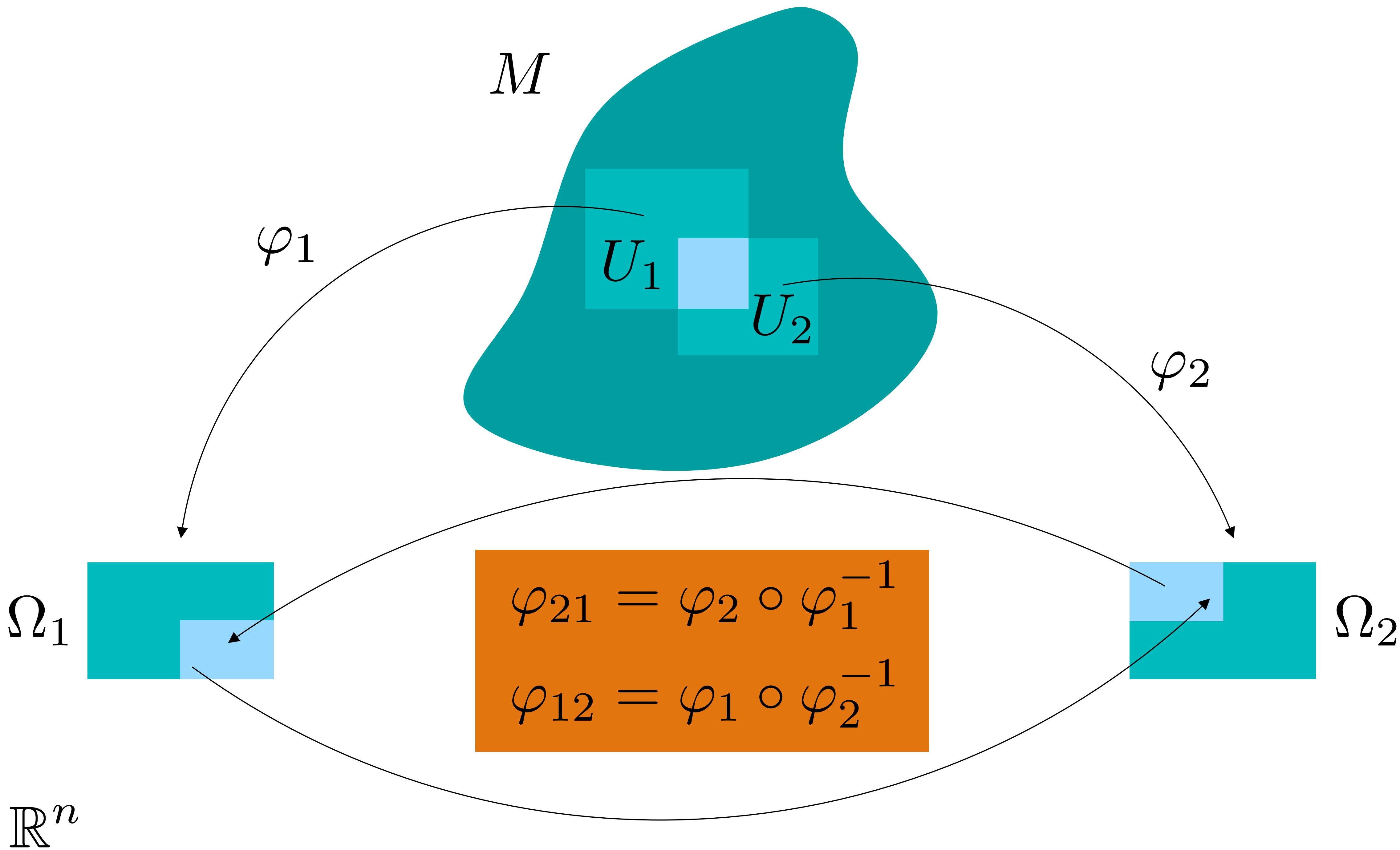

φ_{21} and φ_{12} are the **transition functions.**

Manifolds: Formal Definition

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A C^k n-**atlas** is a family of charts, $\{(U_i, \varphi_i)\}_{(i \in I)}$, where I is a non-empty countable set, and such that the following conditions hold:

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(3) Whenever $U_i \cap U_j \neq \emptyset$, the transition function φ_{ji} (resp. φ_{ij}) is a C^k diffeomorphism.

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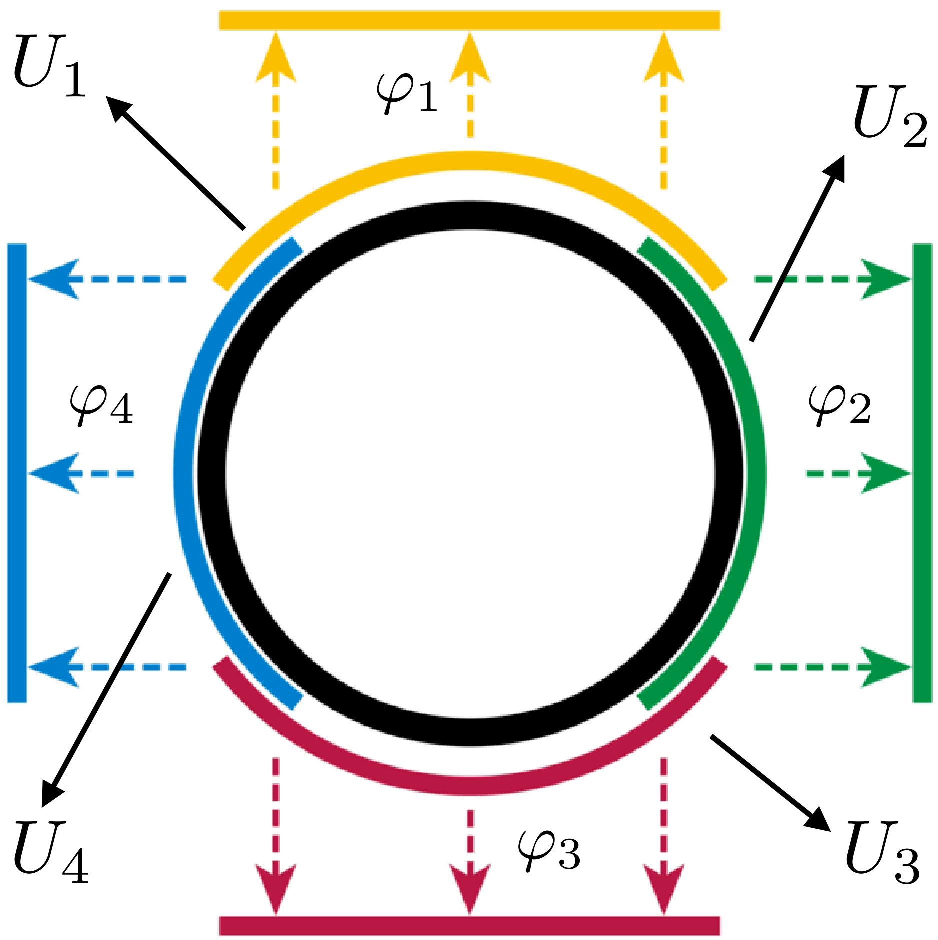

Atlas: $\{(U_1, \varphi_1), (U_2, \varphi_2), (U_3, \varphi_3), (U_4, \varphi_4)\}$

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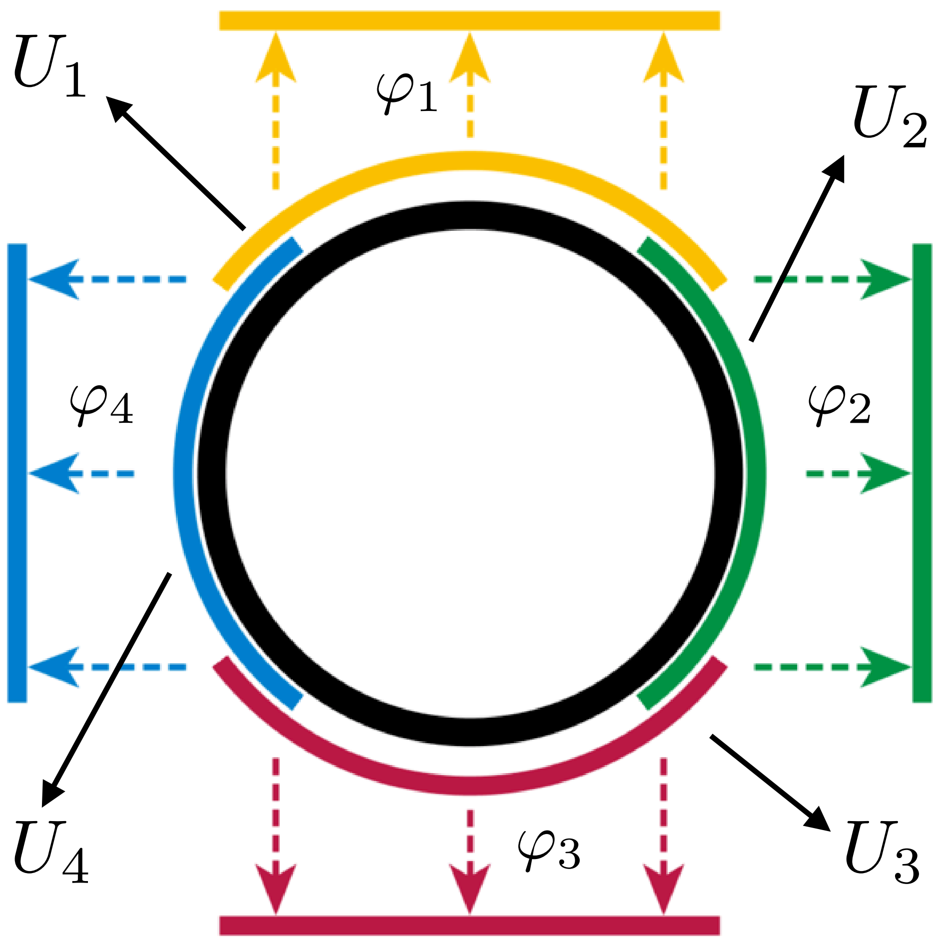

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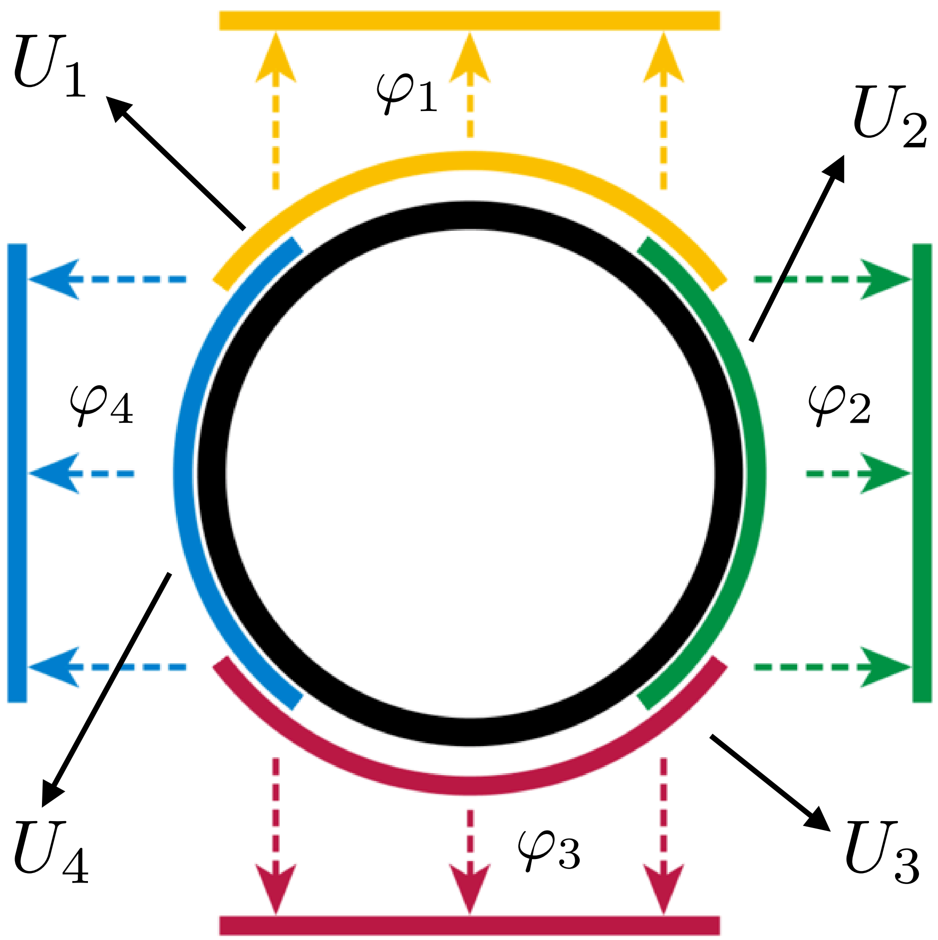

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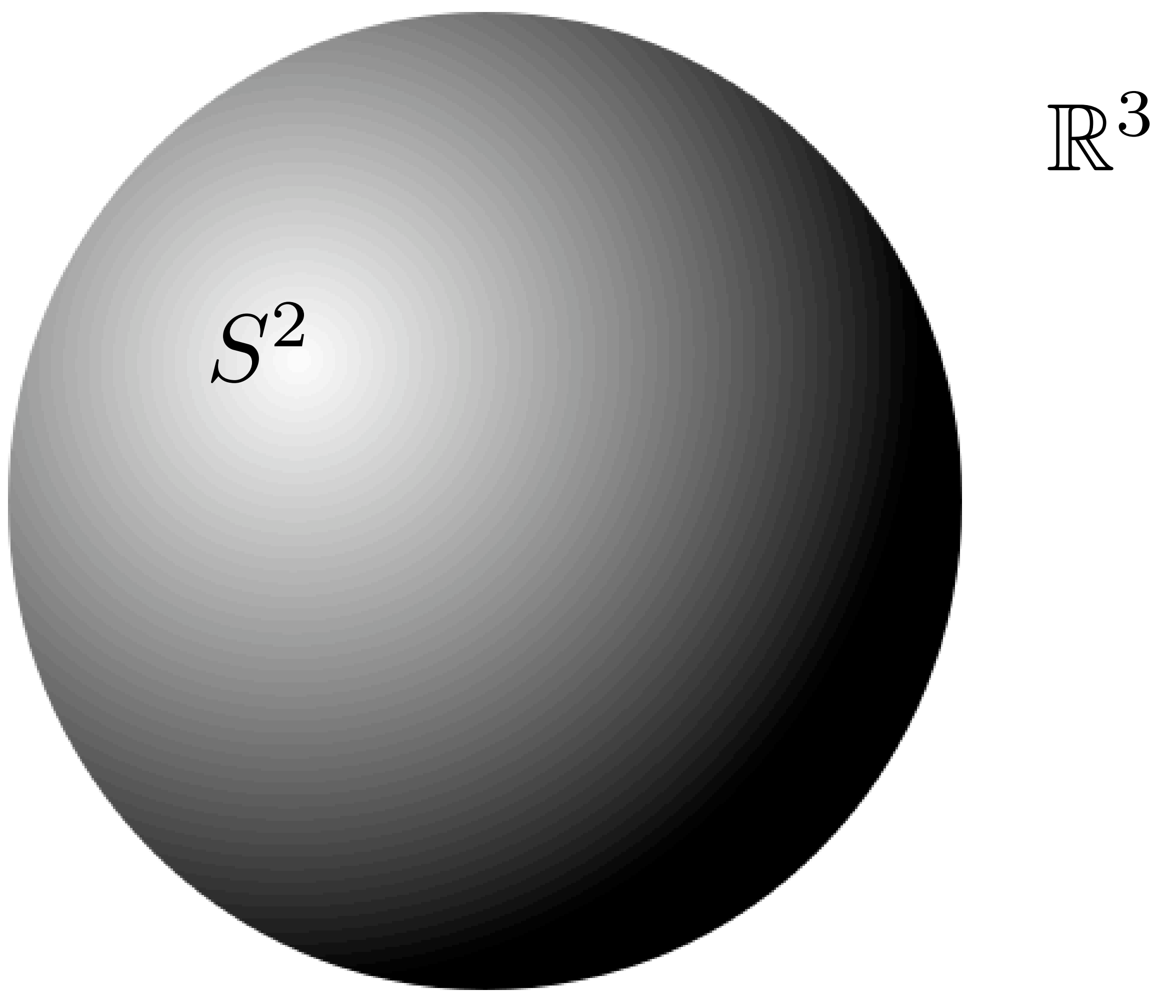
To avoid pathological cases and to ensure that a manifold is always embeddable in $\mathbb{R}^n$, for some $n \geq 1$, we further require that the topology of M be **Hausdorff** and **second-countable**.

Examples

Examples

- The sphere

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\} .$$



Examples

Examples

- We use stereographic projection from the north pole . . .

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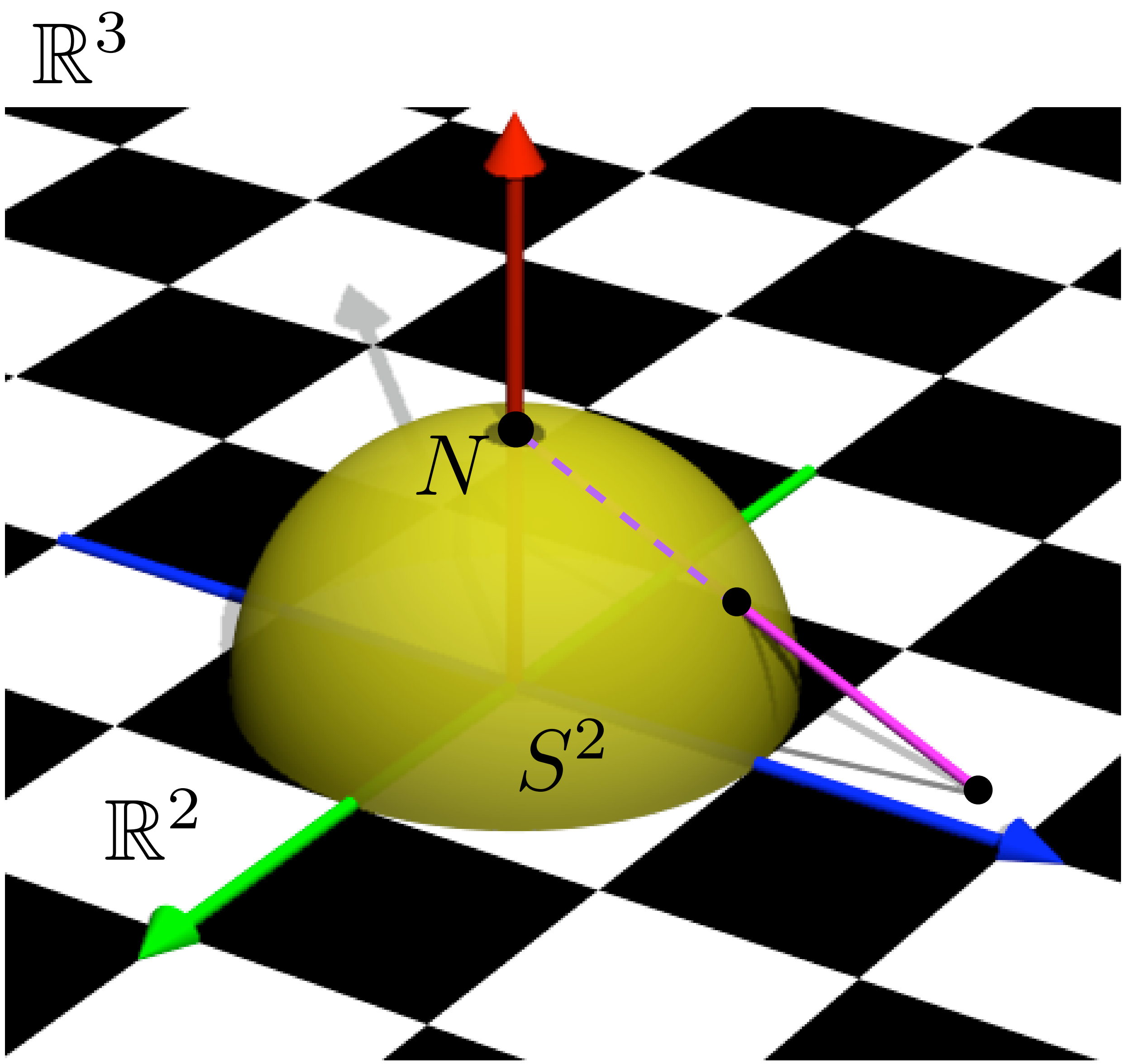

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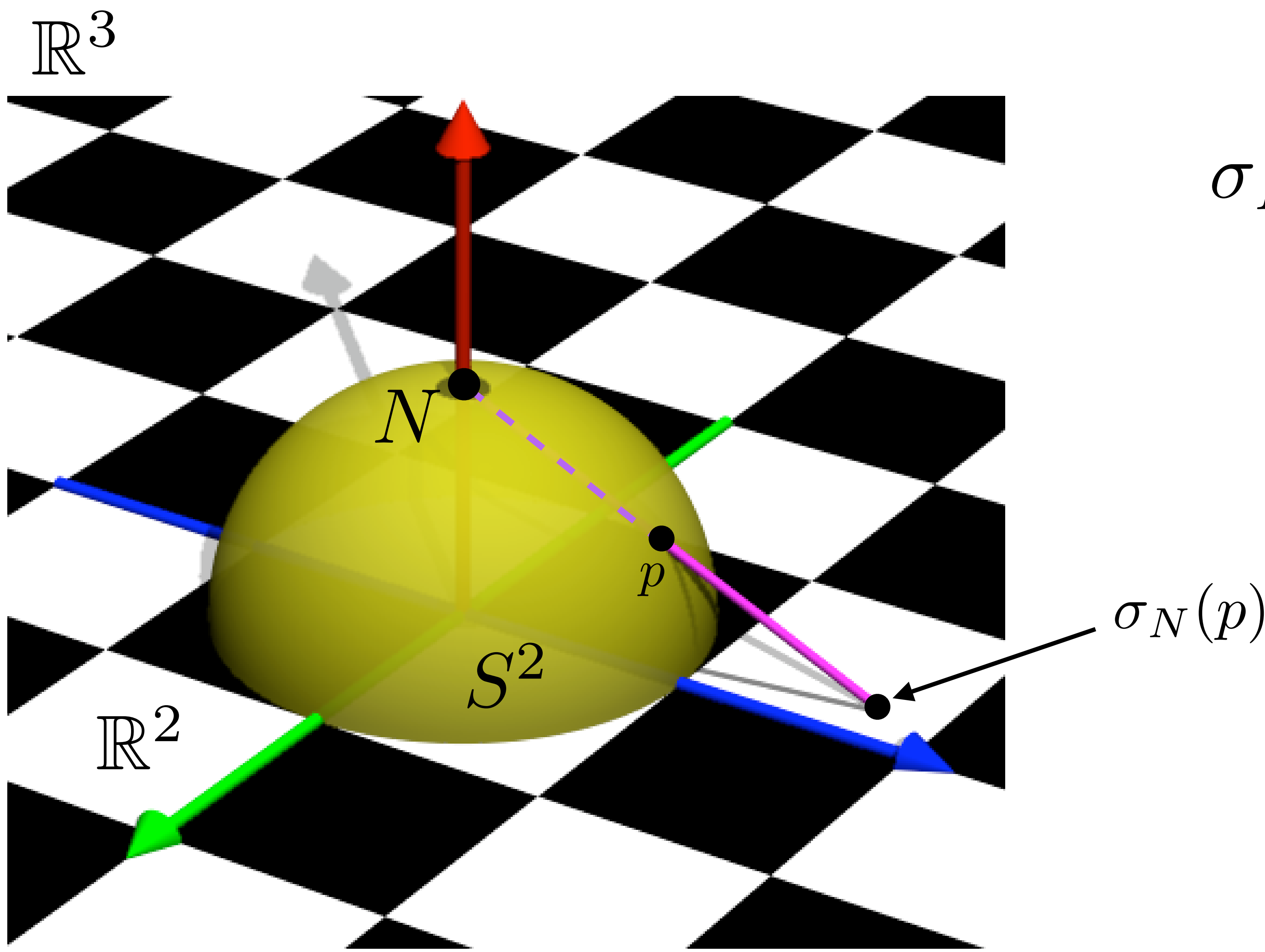

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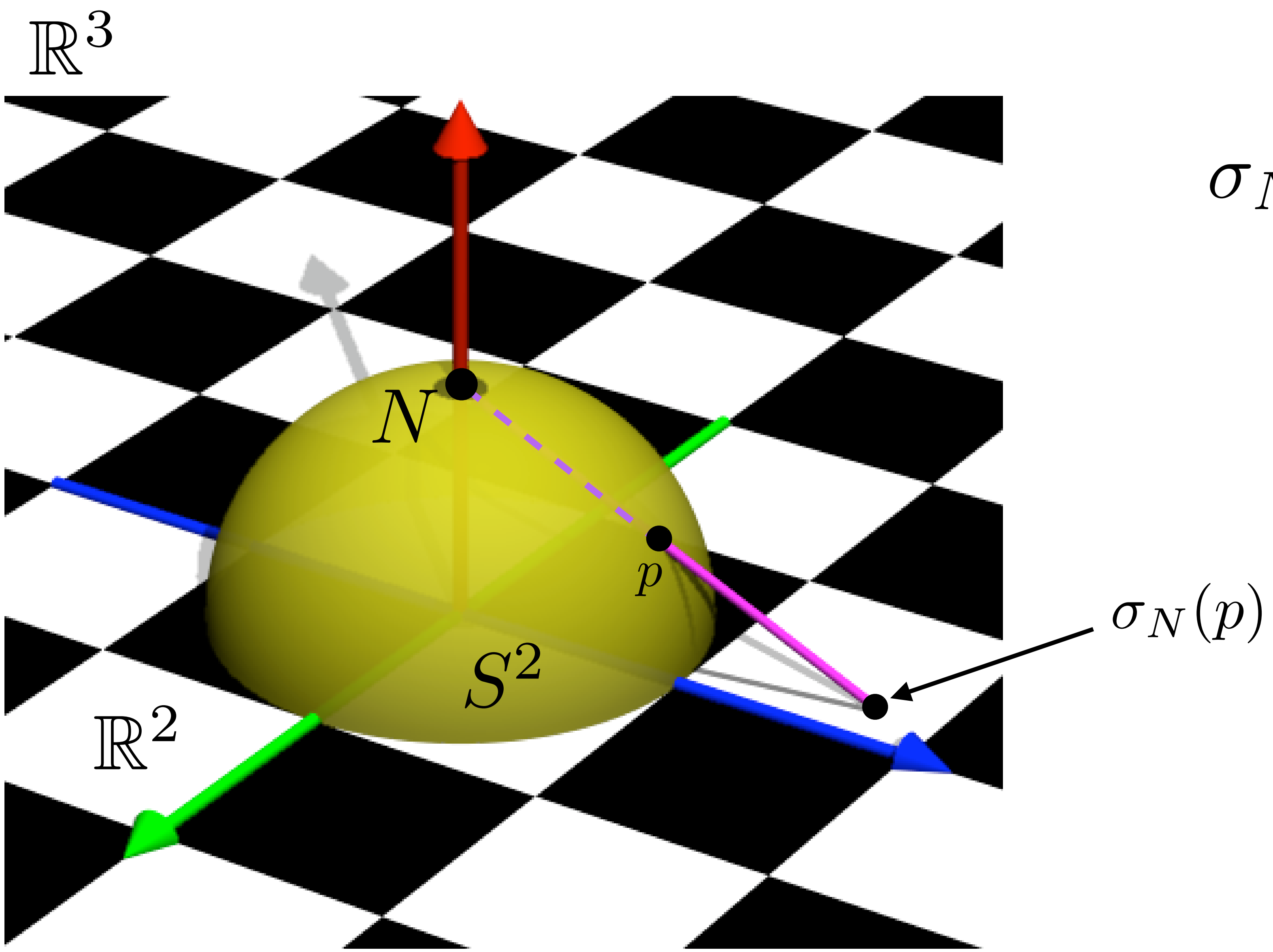

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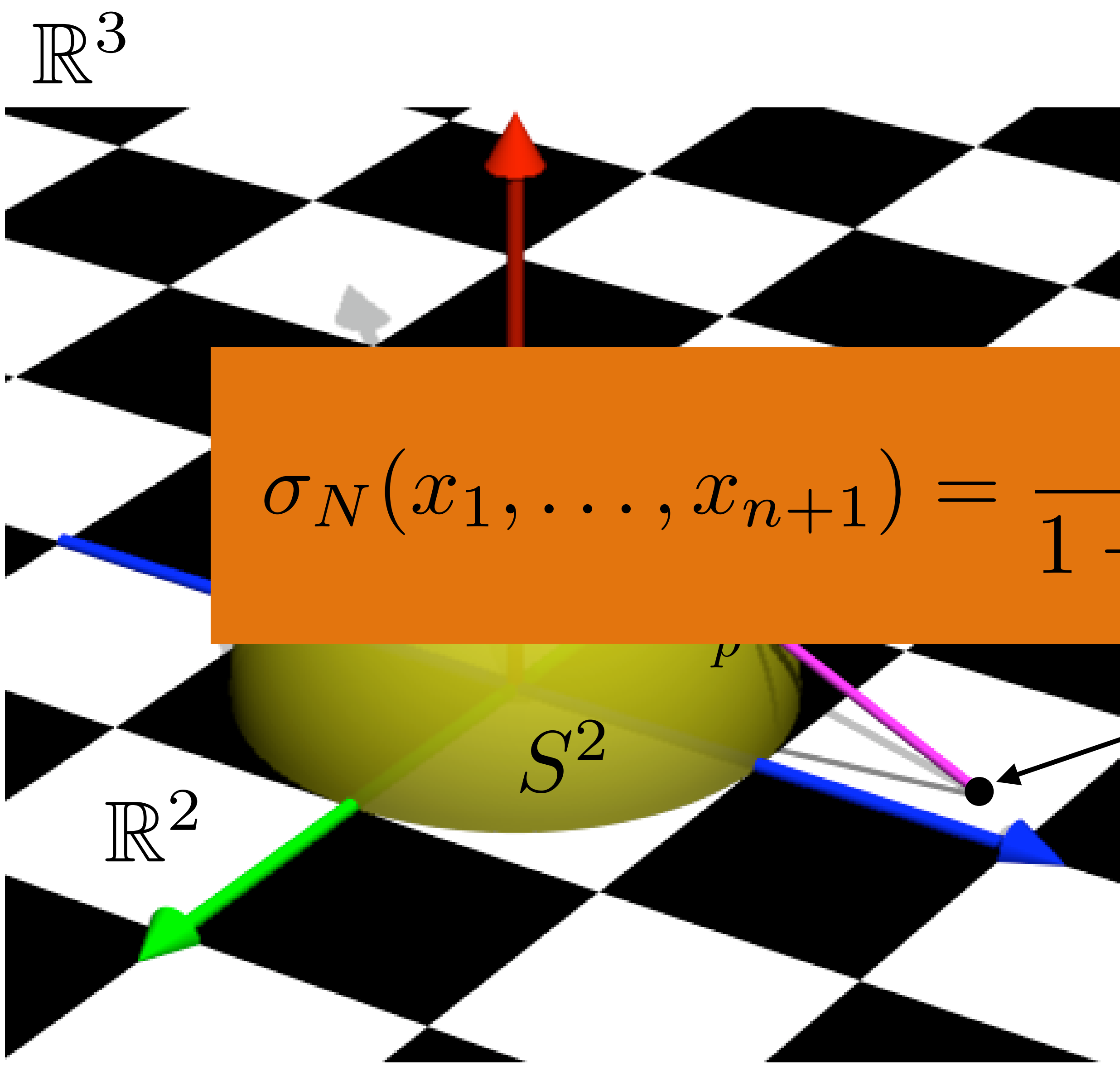

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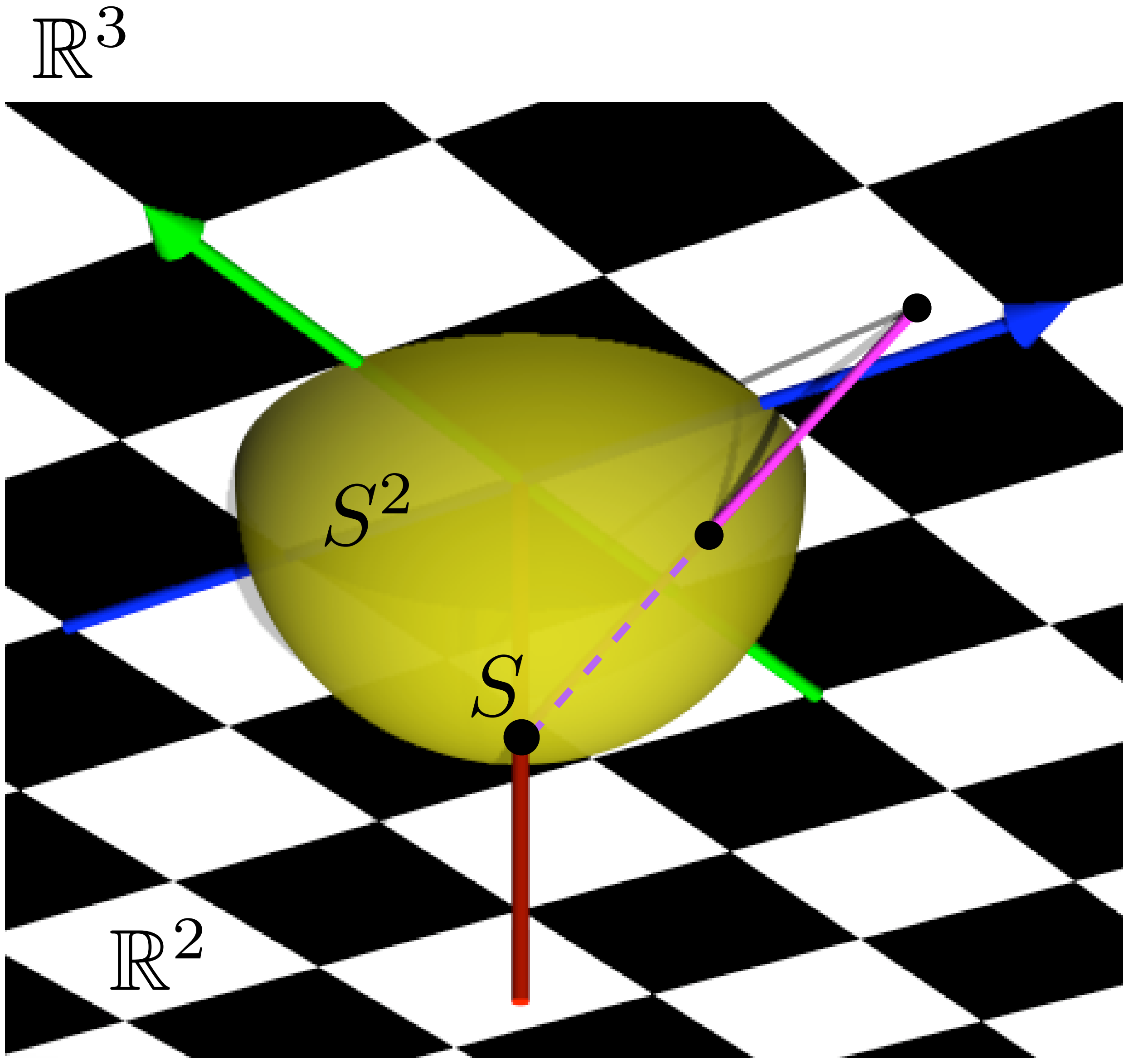

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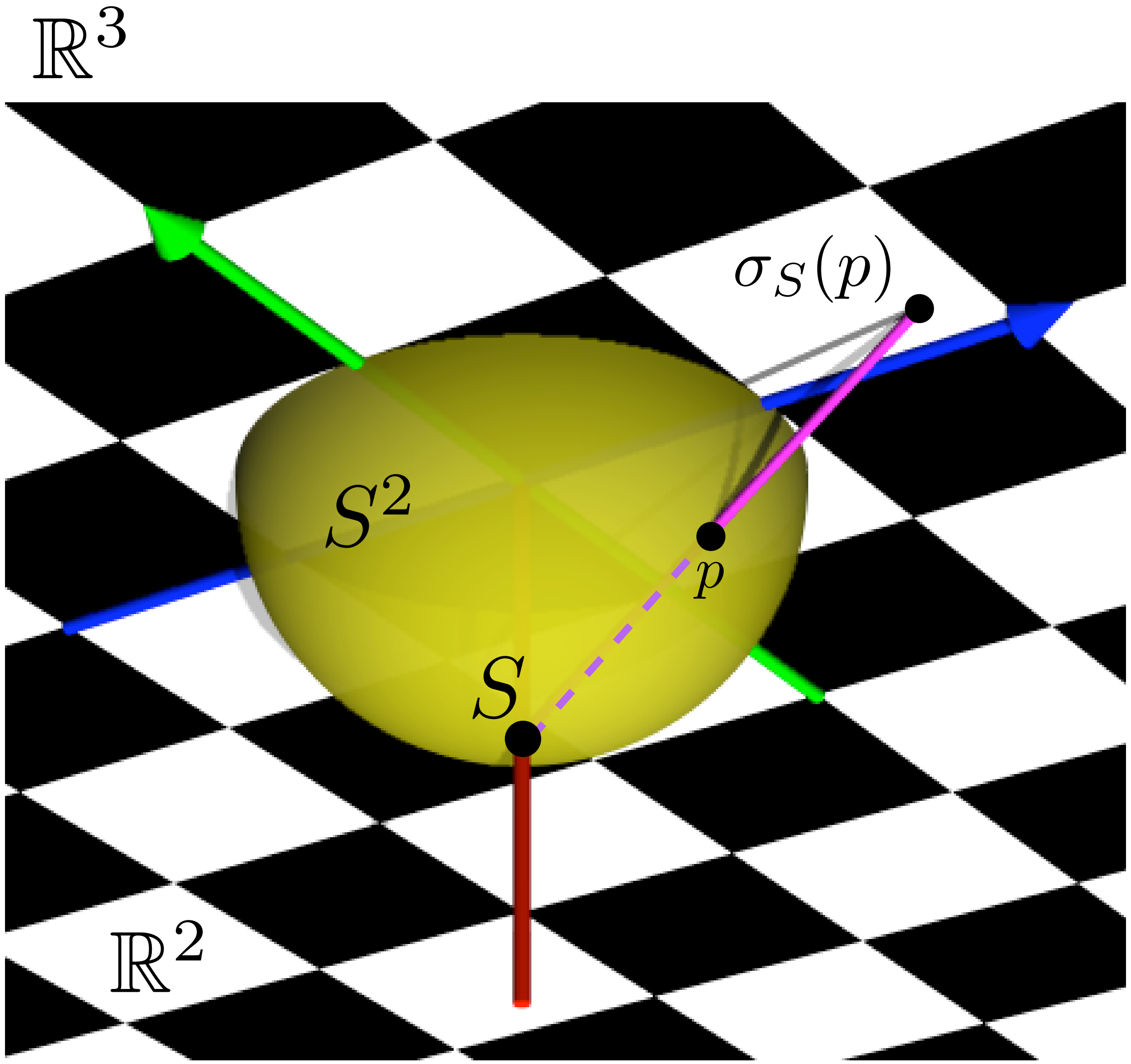

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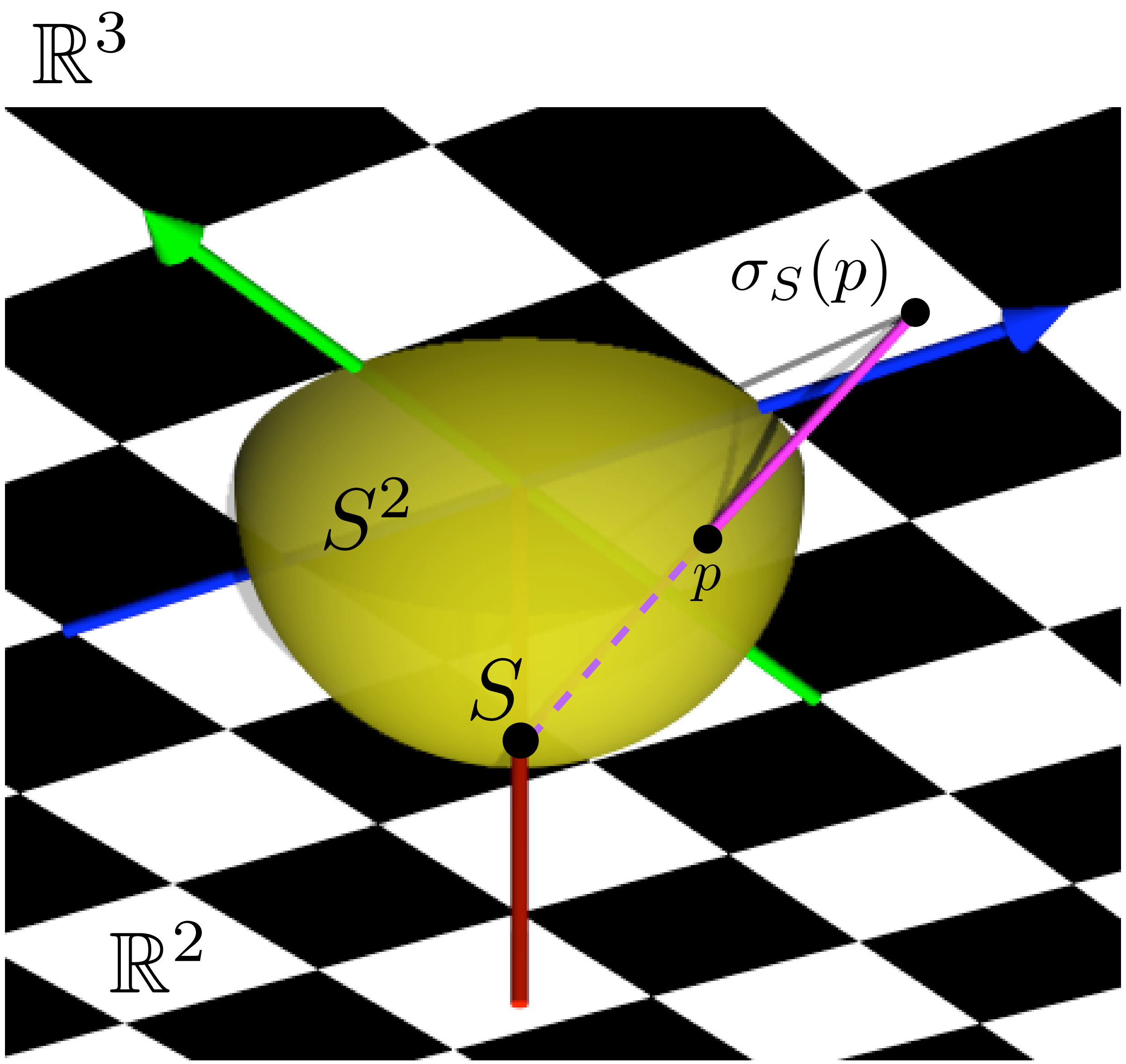

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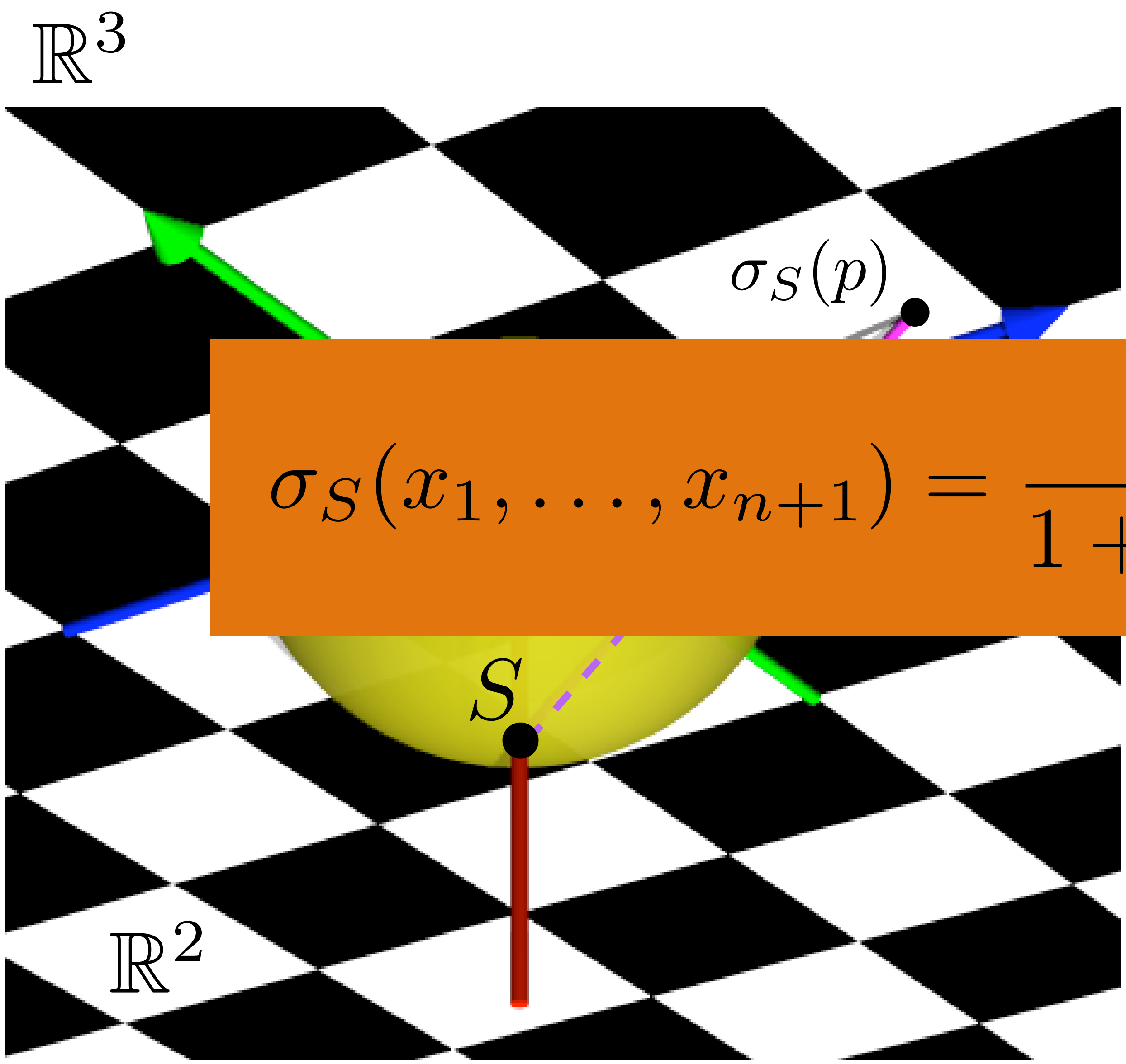

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$$U_N = S^n - \{N\} \qquad \text{and} \qquad U_S = S^n - \{S\}\,.$$

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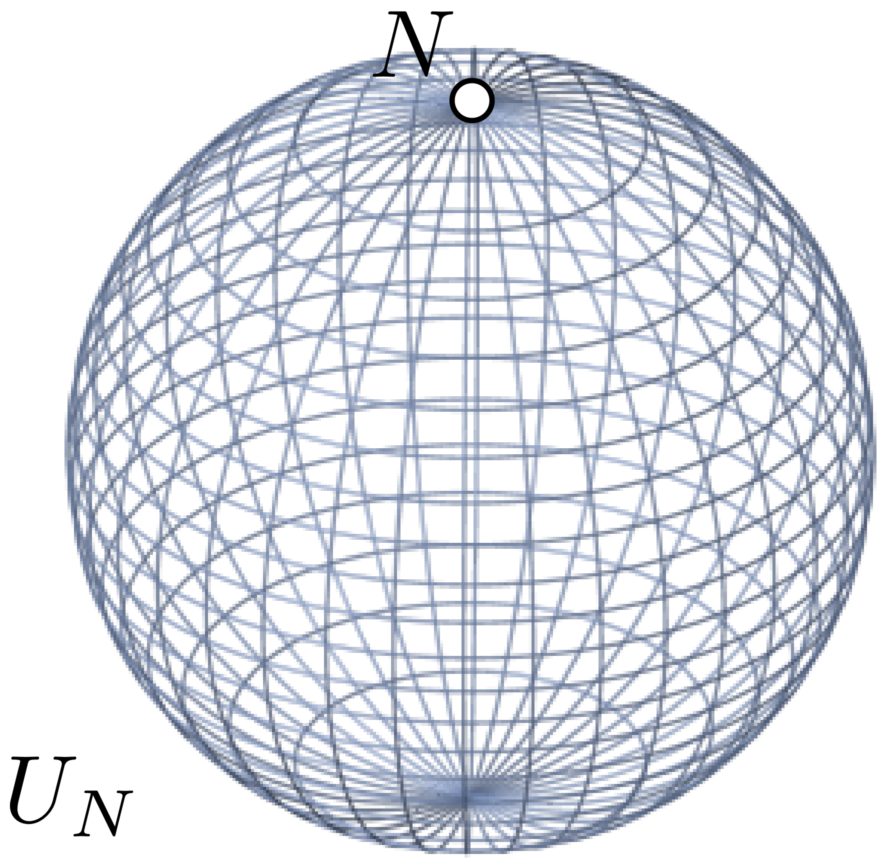

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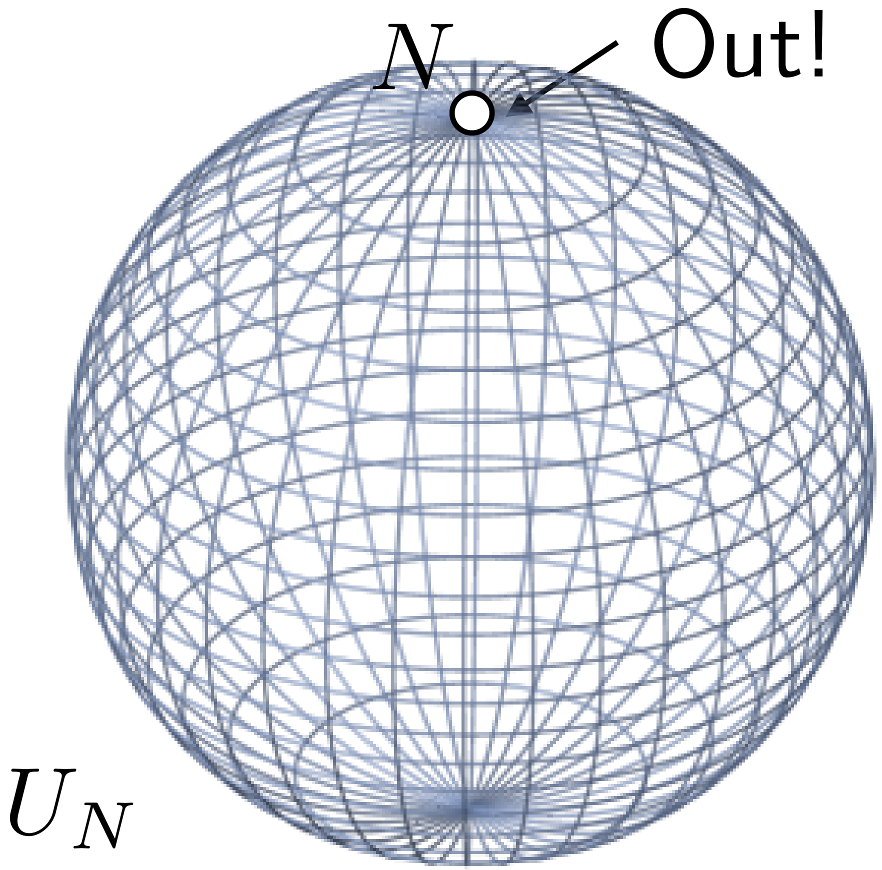

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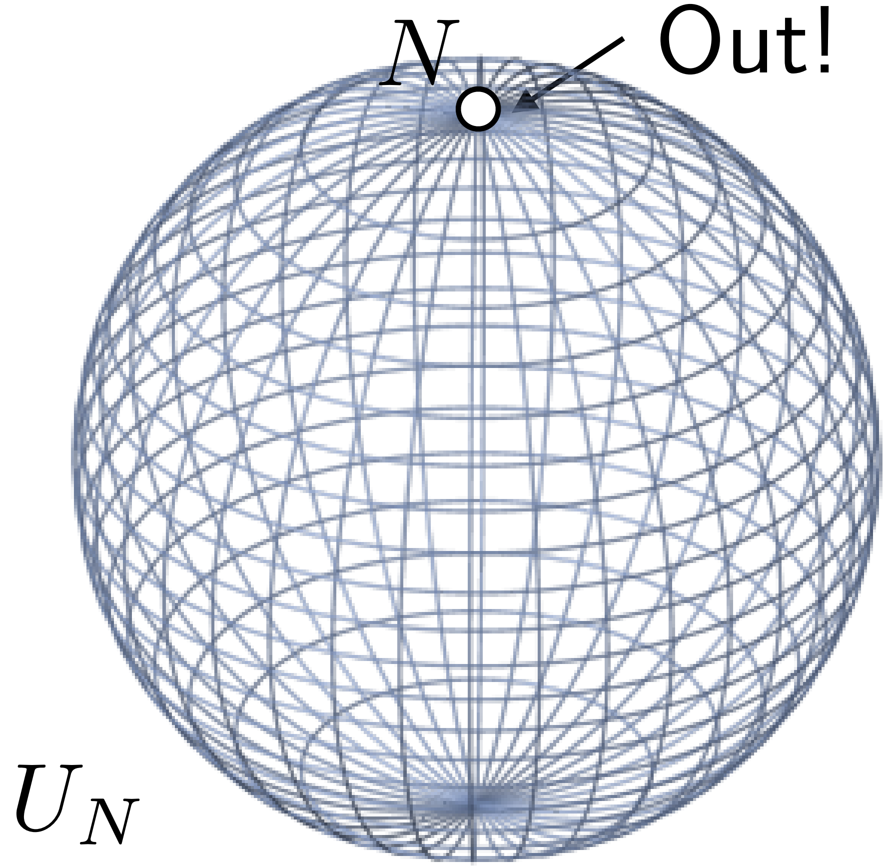

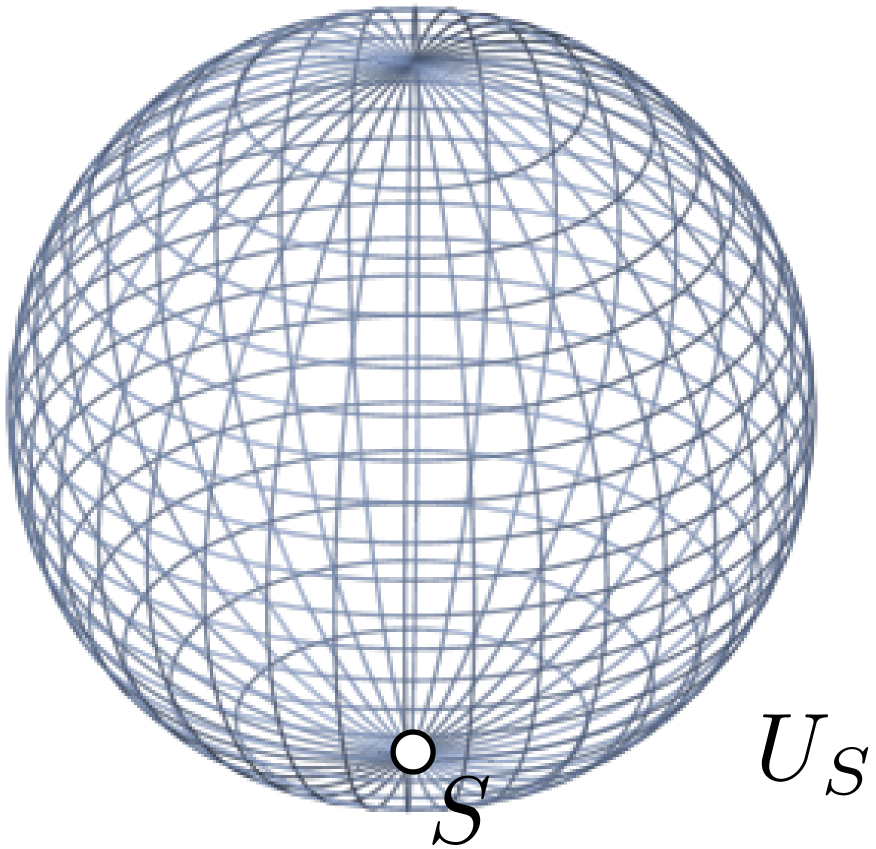

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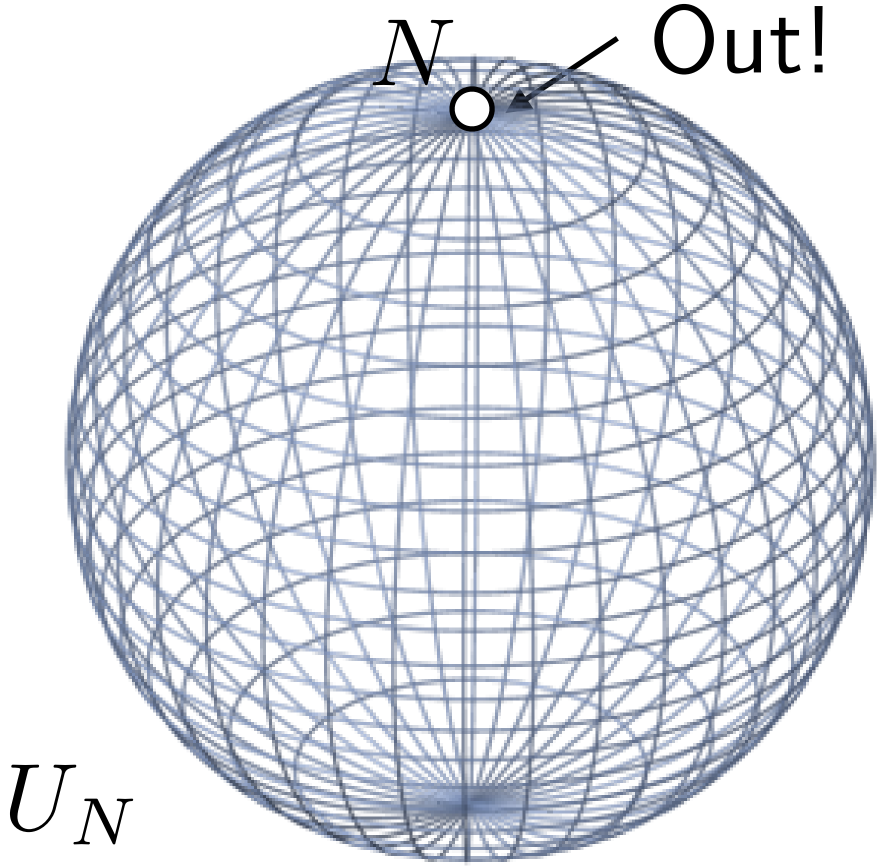

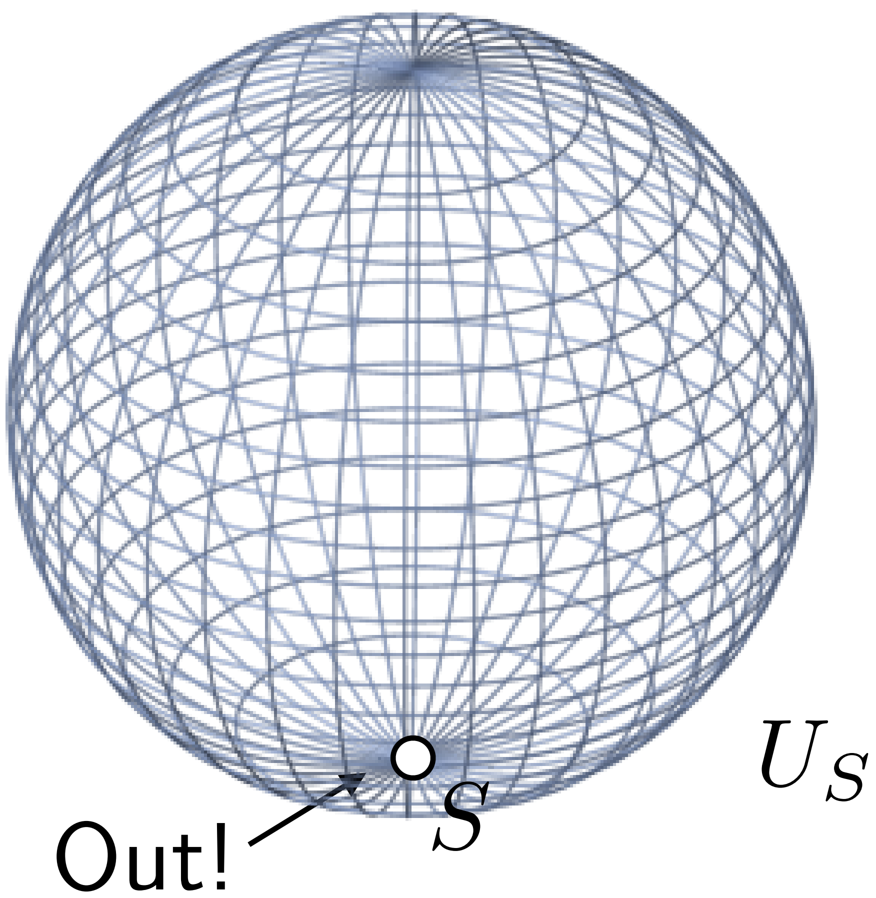

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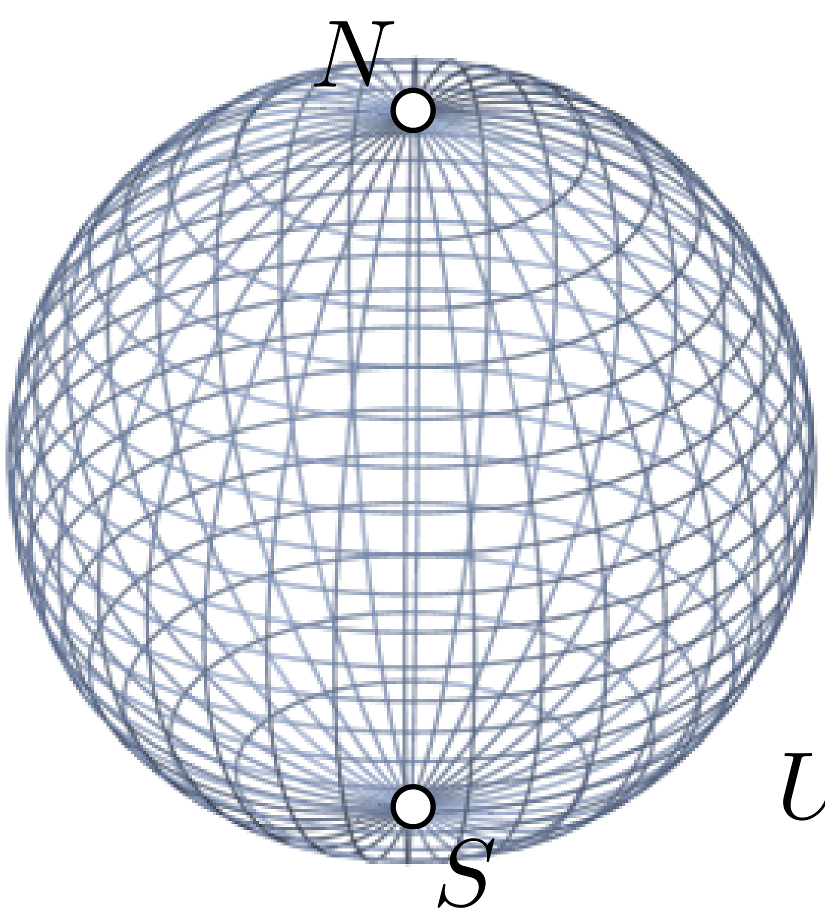

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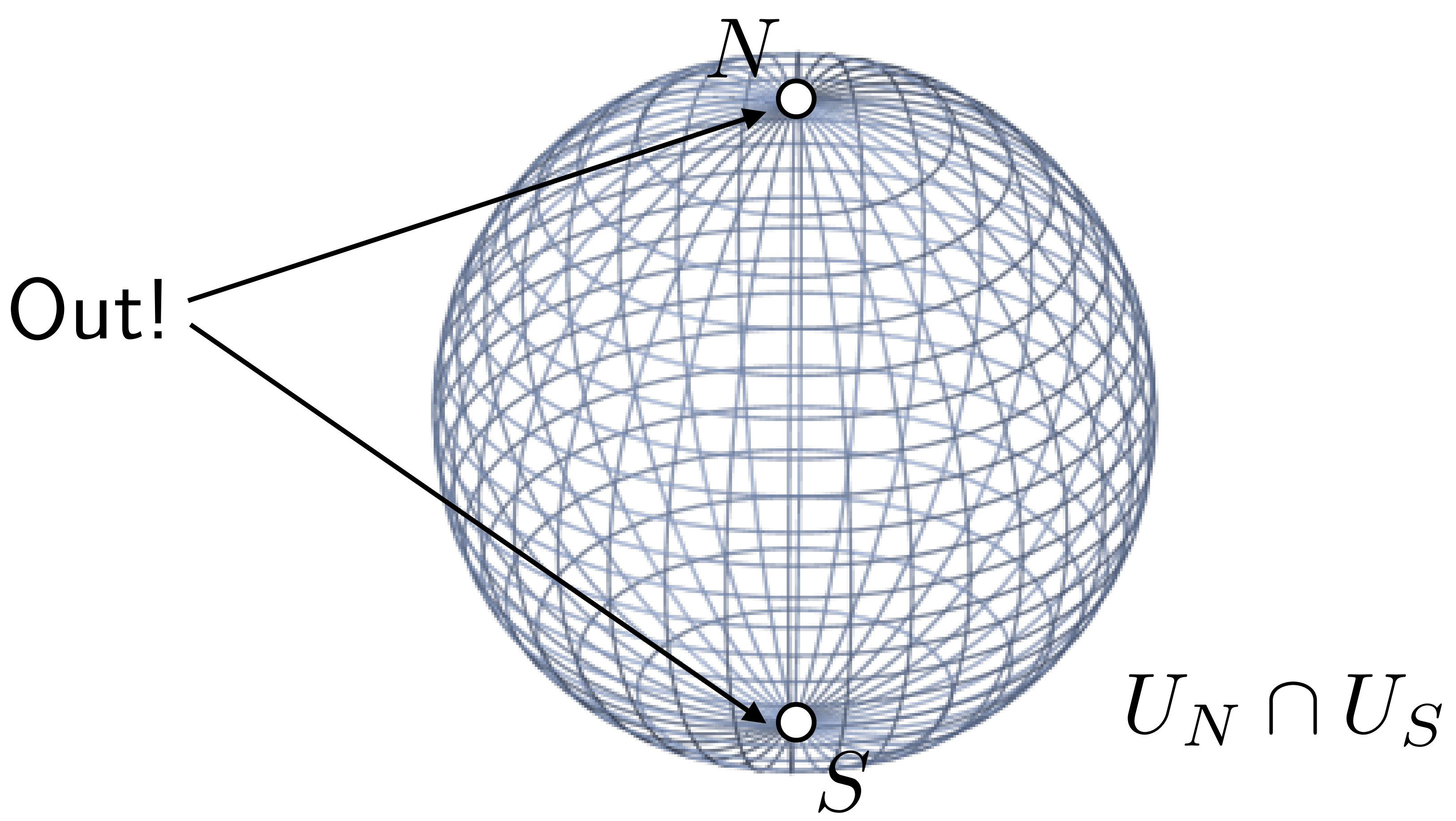

Examples

Examples

- The transition maps

$$\sigma_S \circ \sigma_N^{-1} = \sigma_N \circ \sigma_S^{-1}$$

are given by

$$(x_1, \ldots, x_n) \mapsto \frac{1}{\sum_{i=1}^n x_i^2}(x_1, \ldots, x_n)\,.$$

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- Consequently,

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- So, the sphere is a smooth manifold.

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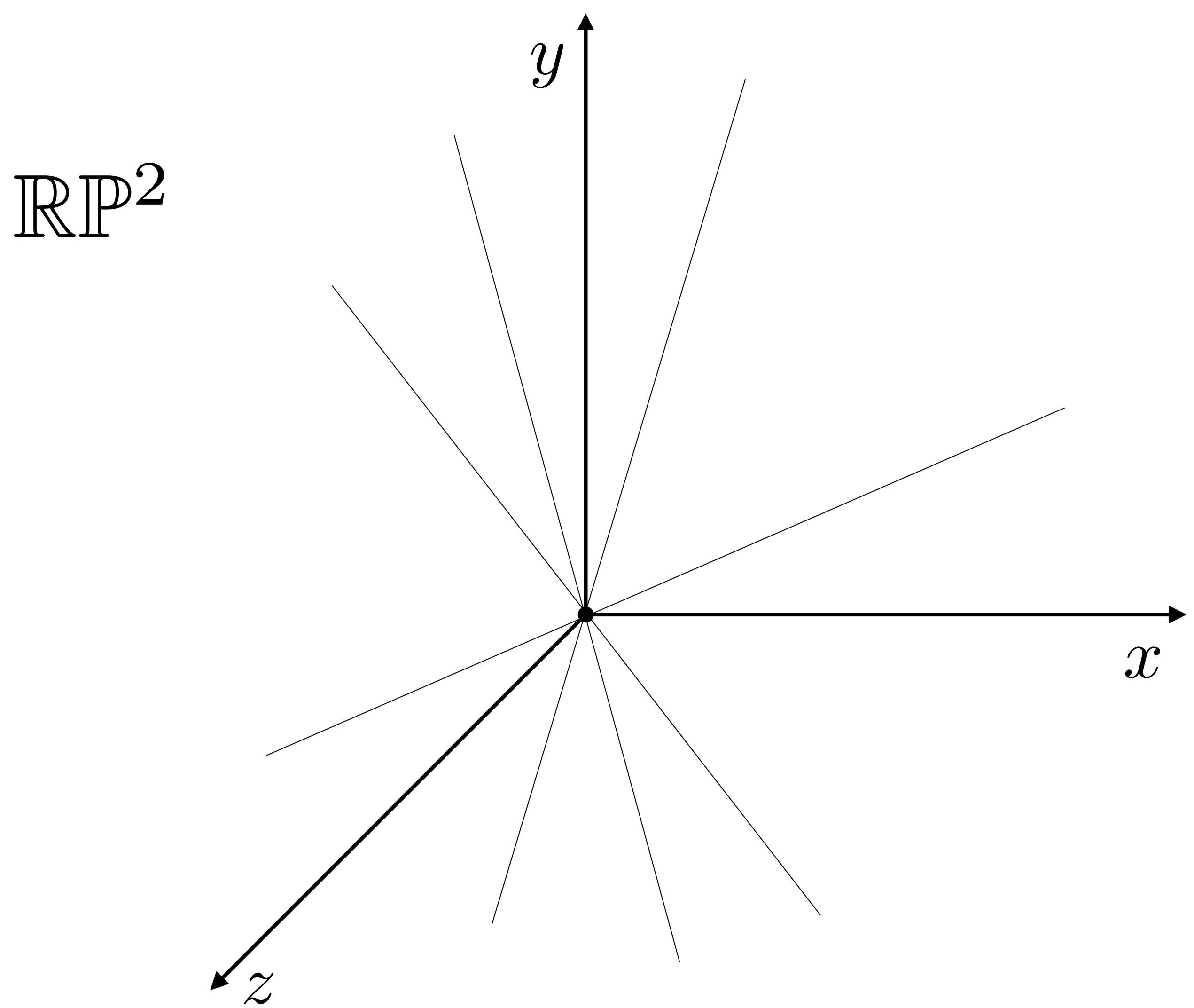

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- Equivalent definition:

  Define an equivalence relation on nonzero vector in $\mathbb{R}^{n+1}$ as follows:

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- Denote the equivalence class of $(x_1, \ldots, x_{n+1})$ by

  $$(x_1 : \cdots : x_{n+1})$$

  also called homogeneous coordinates.

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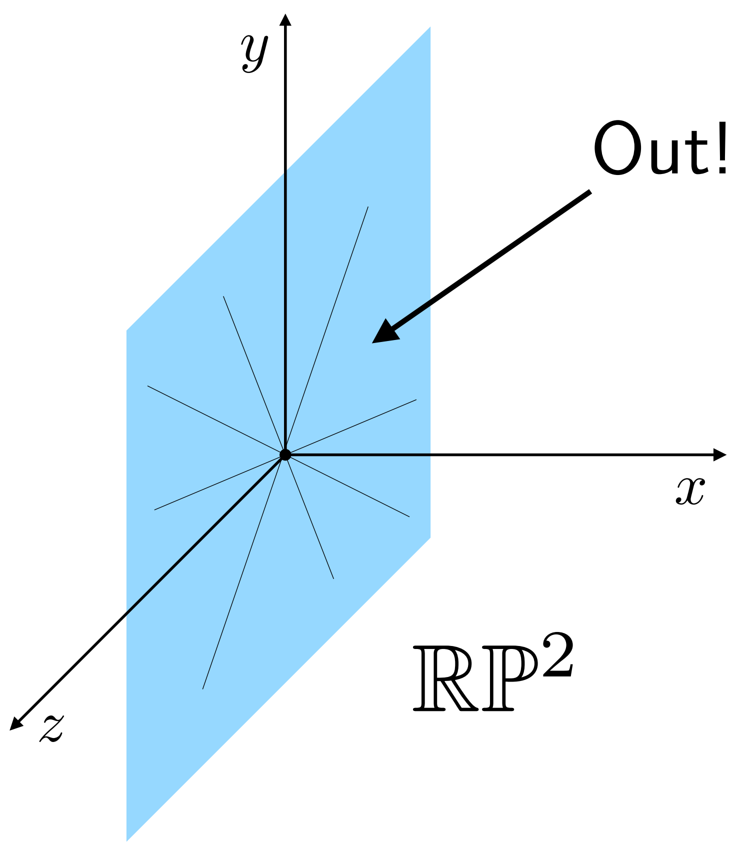

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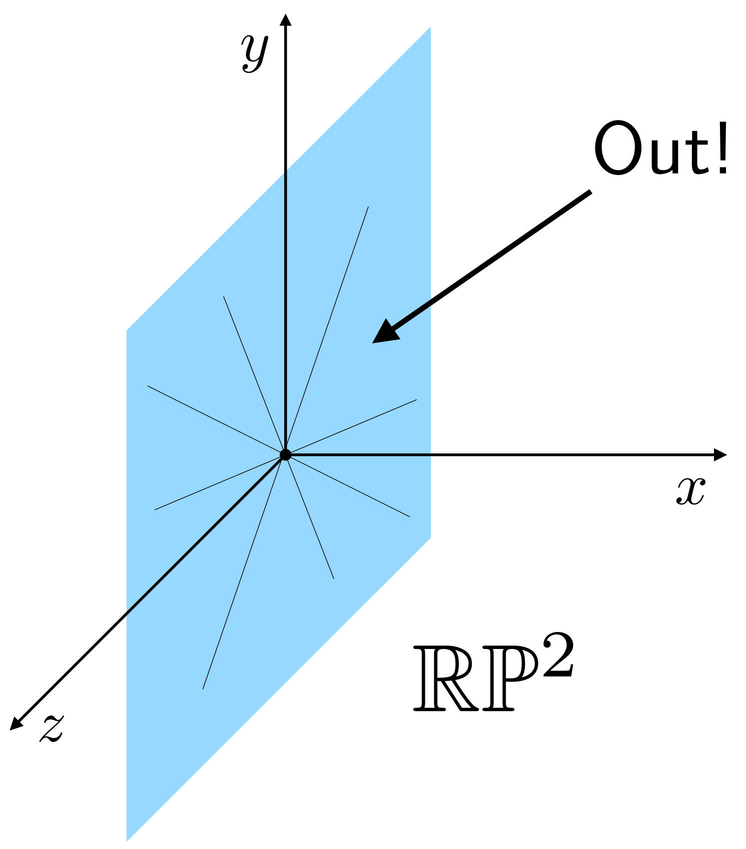

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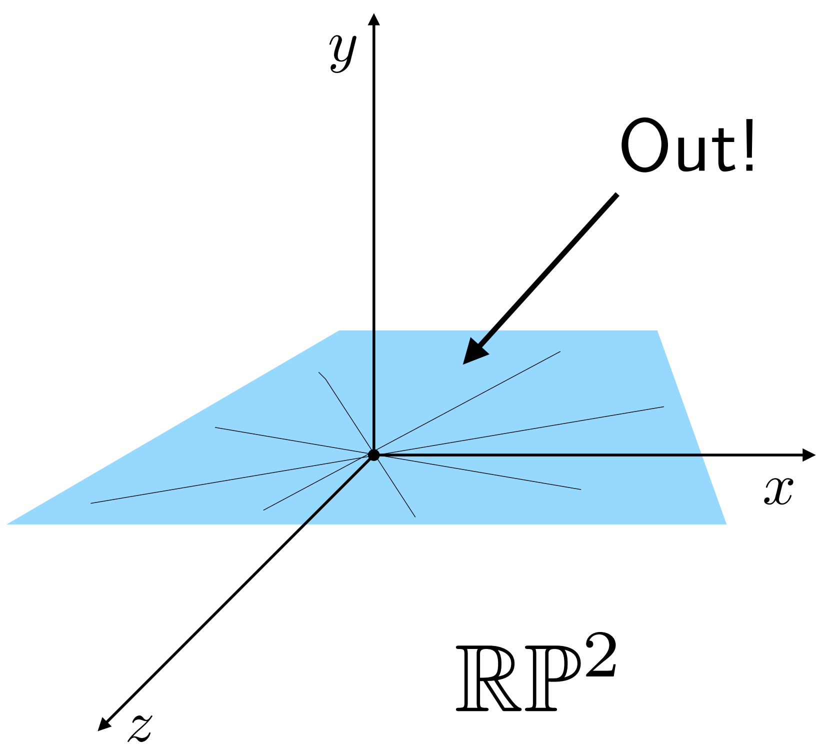

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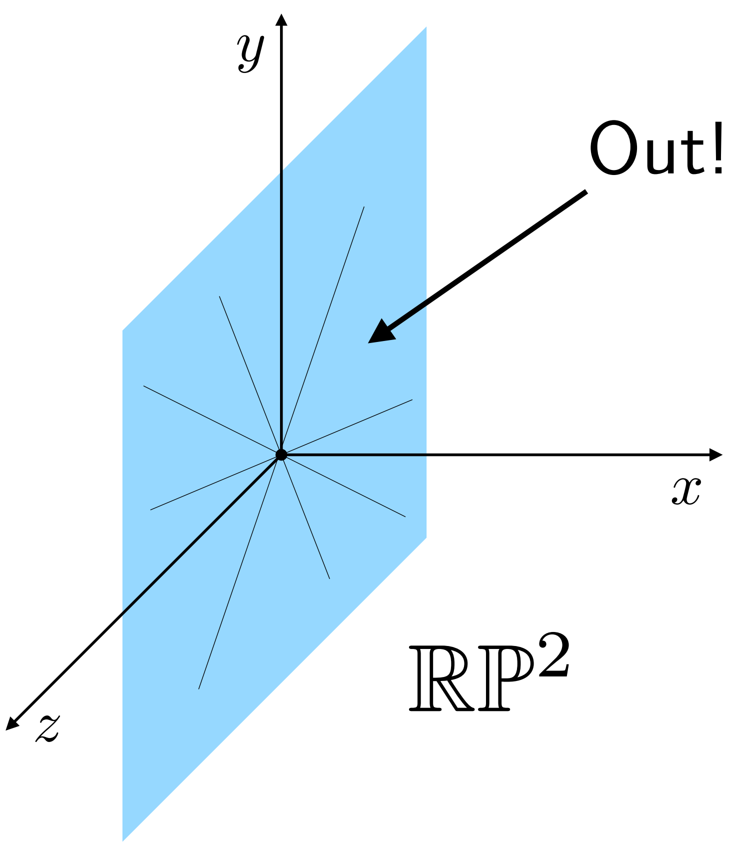

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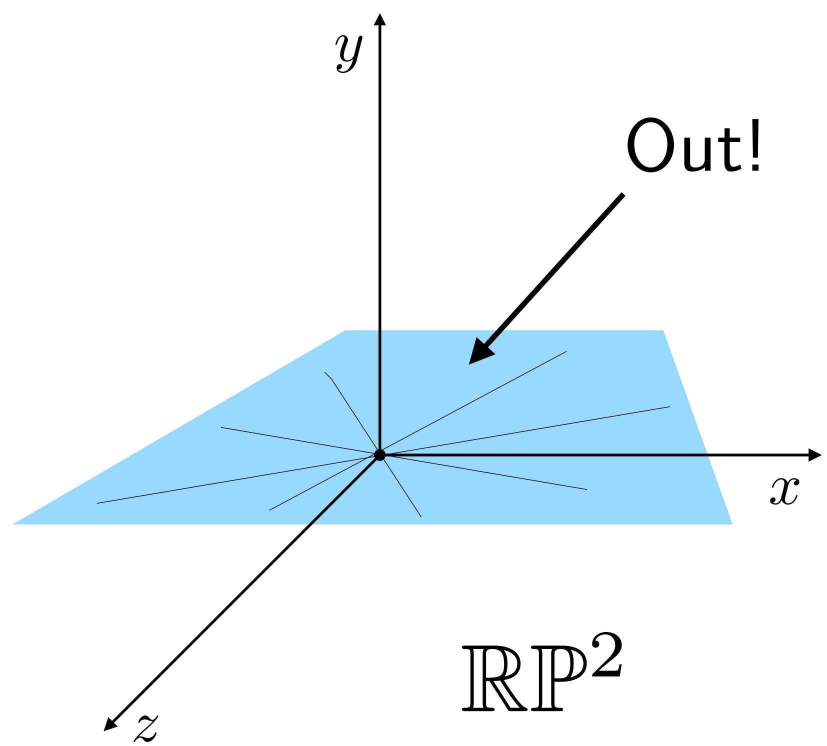

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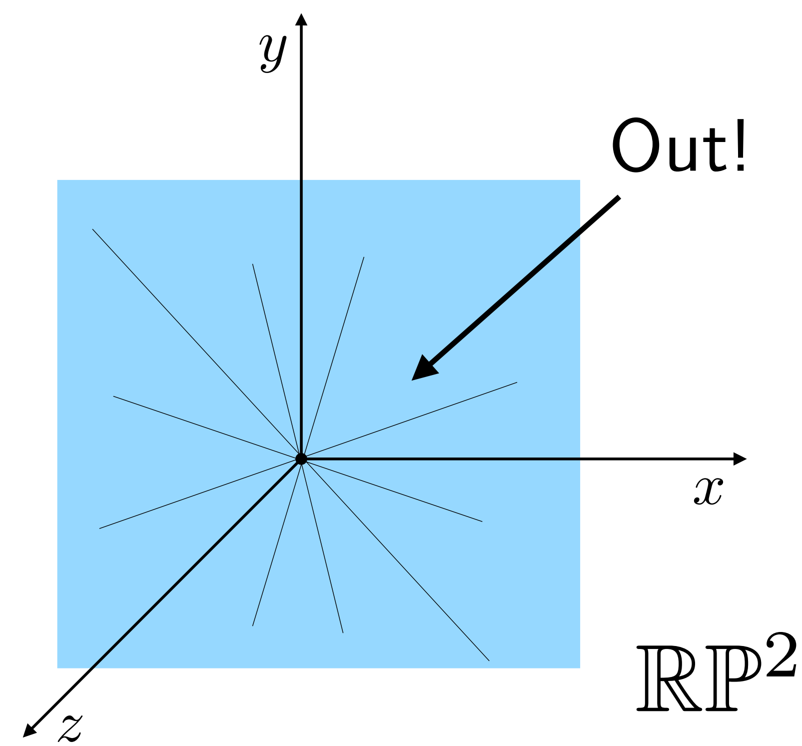

$U_3 = \{(x : y : z) \in \mathbb{RP}^2 \mid z \neq 0\}$

Examples

Examples

- Define $\varphi_i : U_i \to \mathbb{R}^n$ by

$$\varphi_i(x_1 : \cdots : x_{n+1}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right).$$

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$\varphi_2(x : y : z)$

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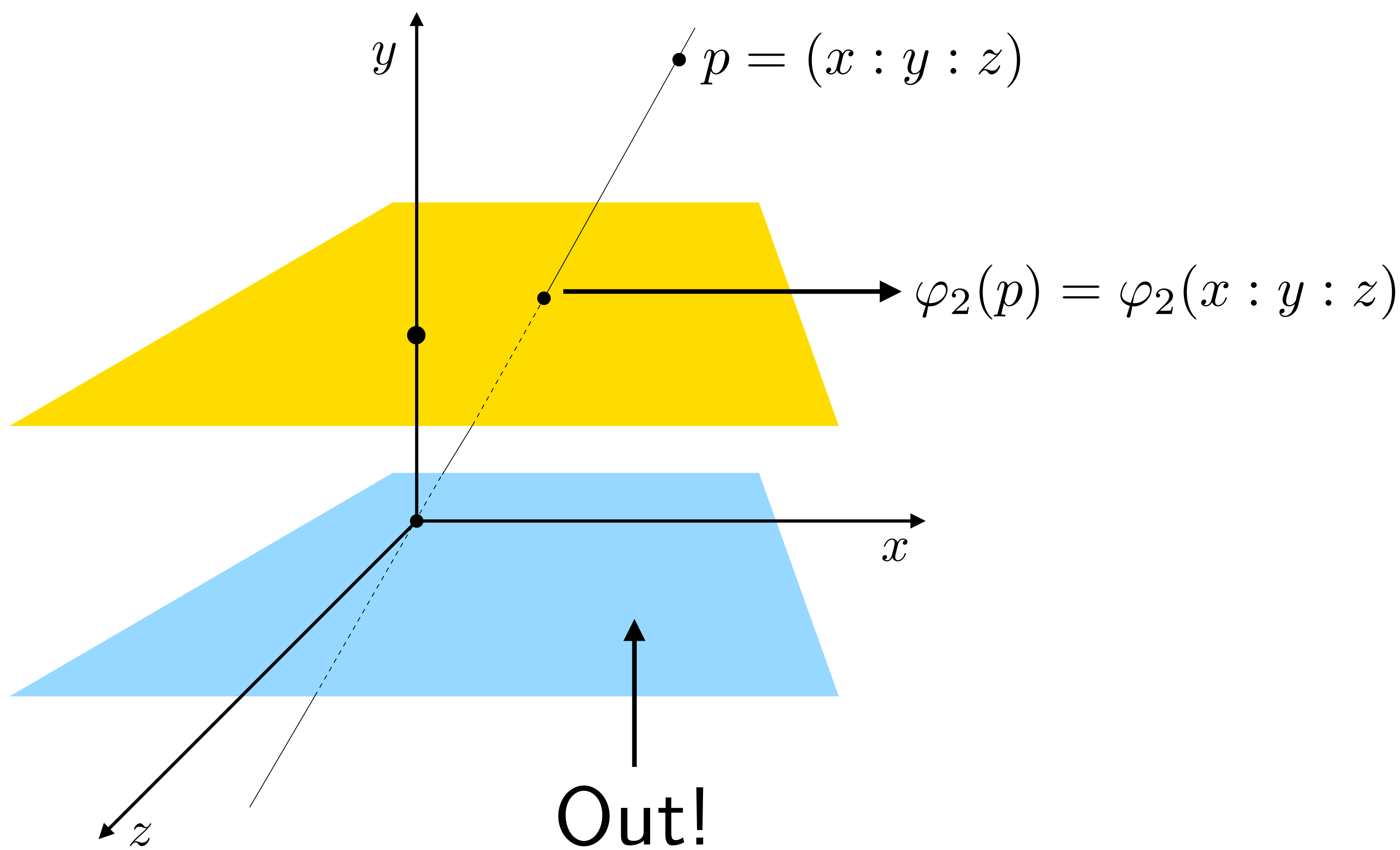

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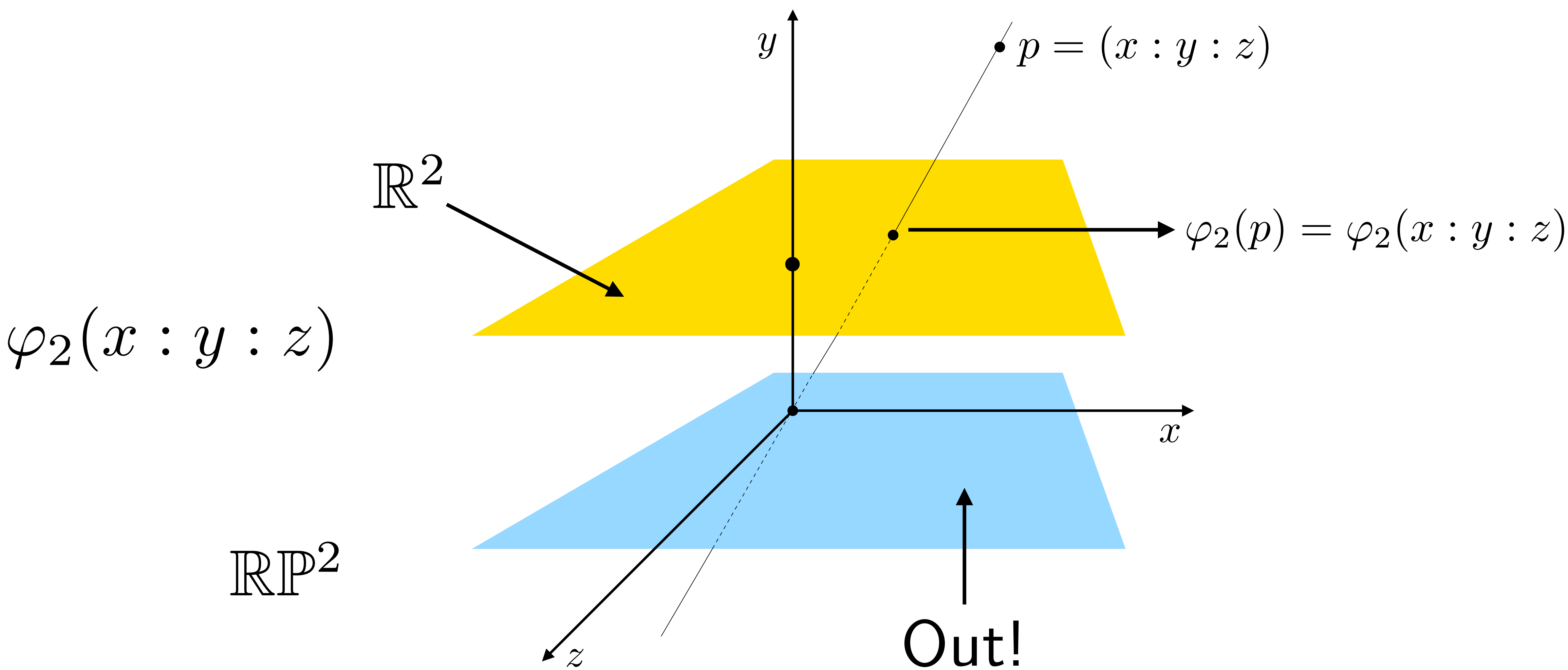

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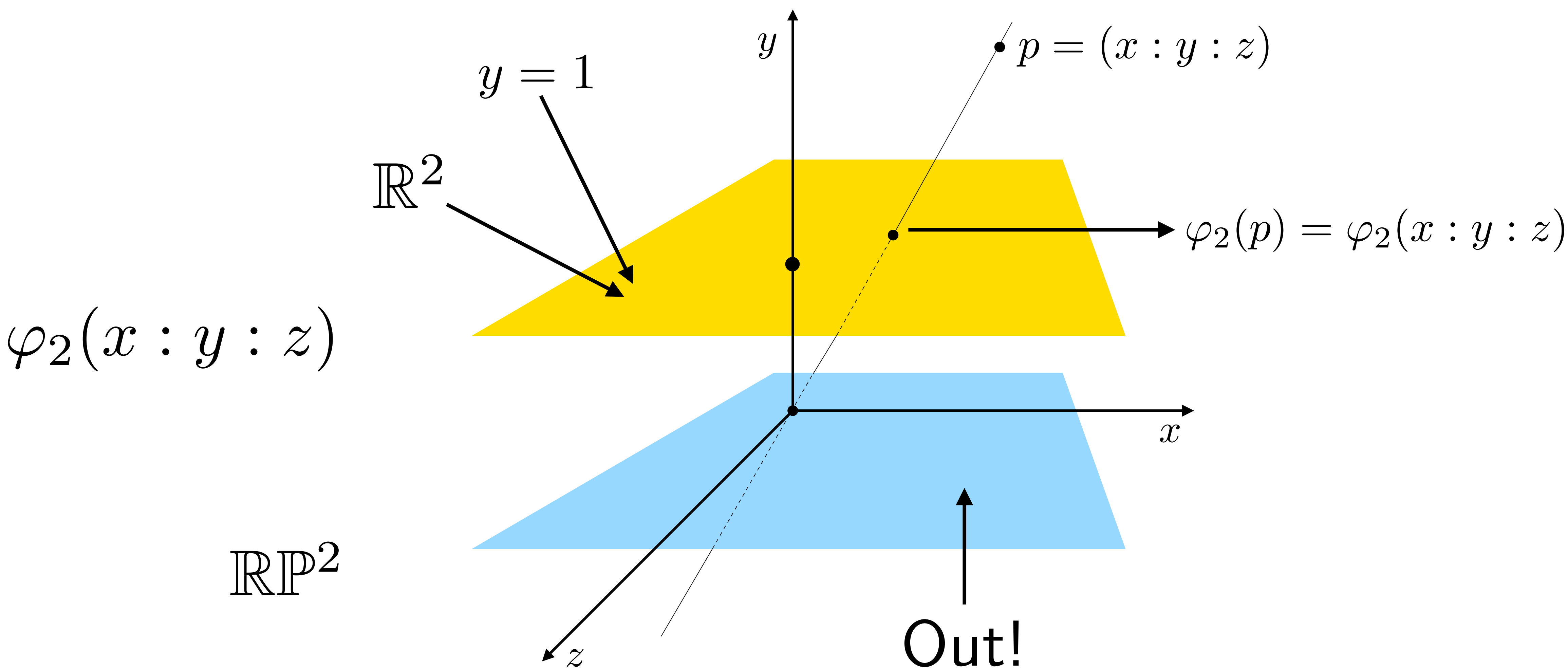
$$\varphi_i(x_1 : \cdots : x_{n+1}) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right) .$$



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- As these maps are smooth, real projective space is a smooth manifold.

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- A similar approach was used to construct fiber bundles in the 1950's (Steenrod).

Constructing Manifolds from Sets of Gluing Data

Jean Gallier
UPenn

Outline

- Motivations
- Sets of gluing data
- Transition functions
- The cocyle condition
- Parametric pseudo manifolds (PPM's)
- Conclusions

Motivations

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- More specifically, we want to build a C^k two-dimensional manifold in $\mathbb{R}^3$.

- Our plan is to define S constructively by **building a manifold**.

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A LITTLE PROBLEM:

Our definition of manifold is not constructive: it states what a manifold is by assuming it already exists! So, for our purposes, it is not useful!

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THE KEY IDEA:

The notion of a **set of gluing data**.

Sets of Gluing Data

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Let n and k be integers such that $n \geq 1$ and $k \geq 1$ (or $k = \infty$).

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A set of gluing data is a triple

$$\mathcal{G} = \big((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ji})_{(i,j) \in K \times K}\big)$$

satisfying the following properties, where I and K are countable sets and I is non-empty:

Sets of Gluing Data

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(1) For every $i \in I$, the set Ω_i is a non-empty open subset of $\mathbb{R}^n$ called **parametrization domain**, for short, p-**domain**, and the Ω_i are pairwise disjoint (i.e., $\Omega_i \cap \Omega_j = \emptyset$ for all $i \neq j$).

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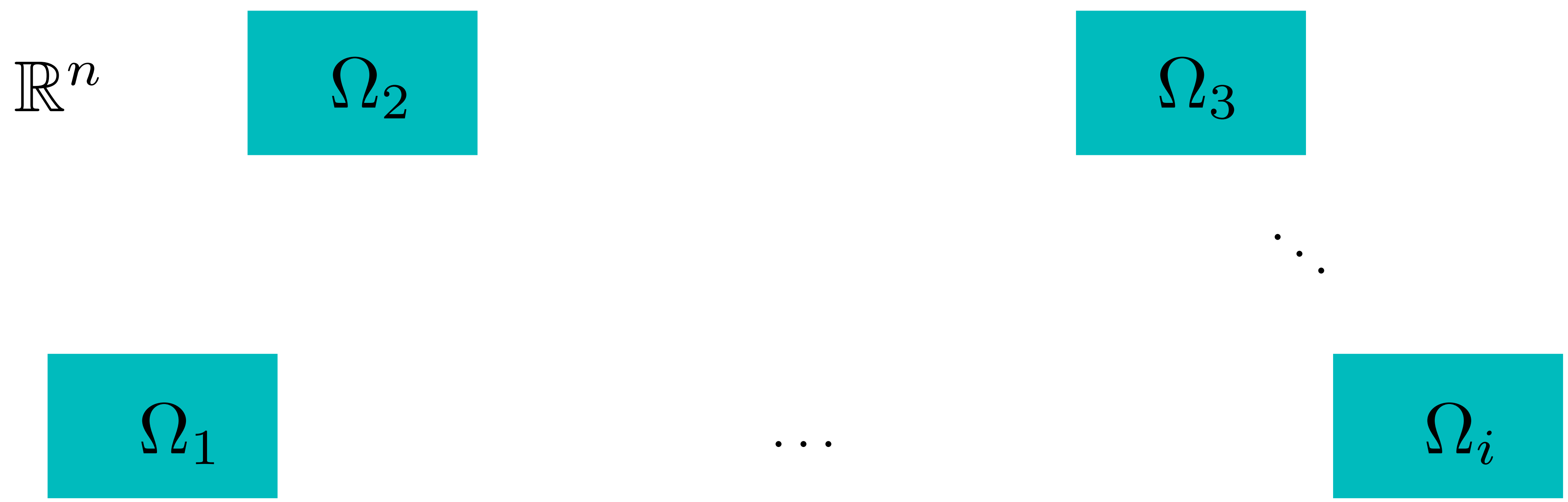

Sets of Gluing Data

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(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$, and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$. Each non-empty Ω_{ij} (with $i \neq j$) is called **gluing domain**.

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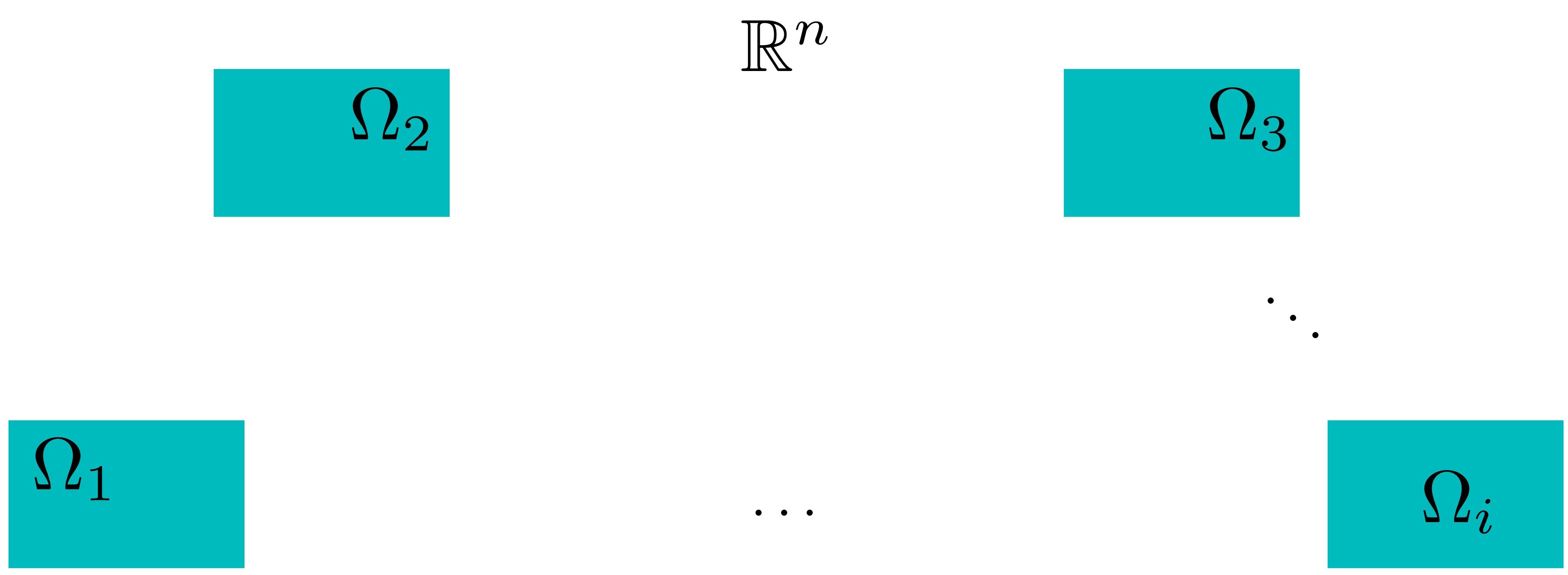

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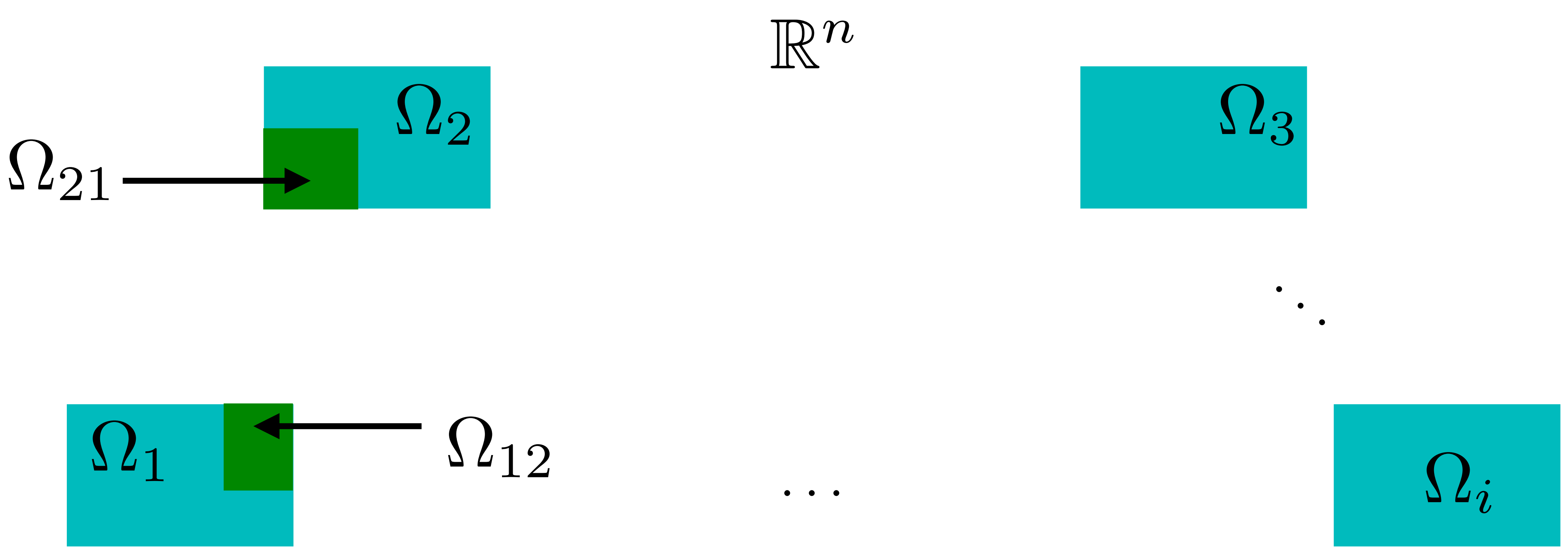

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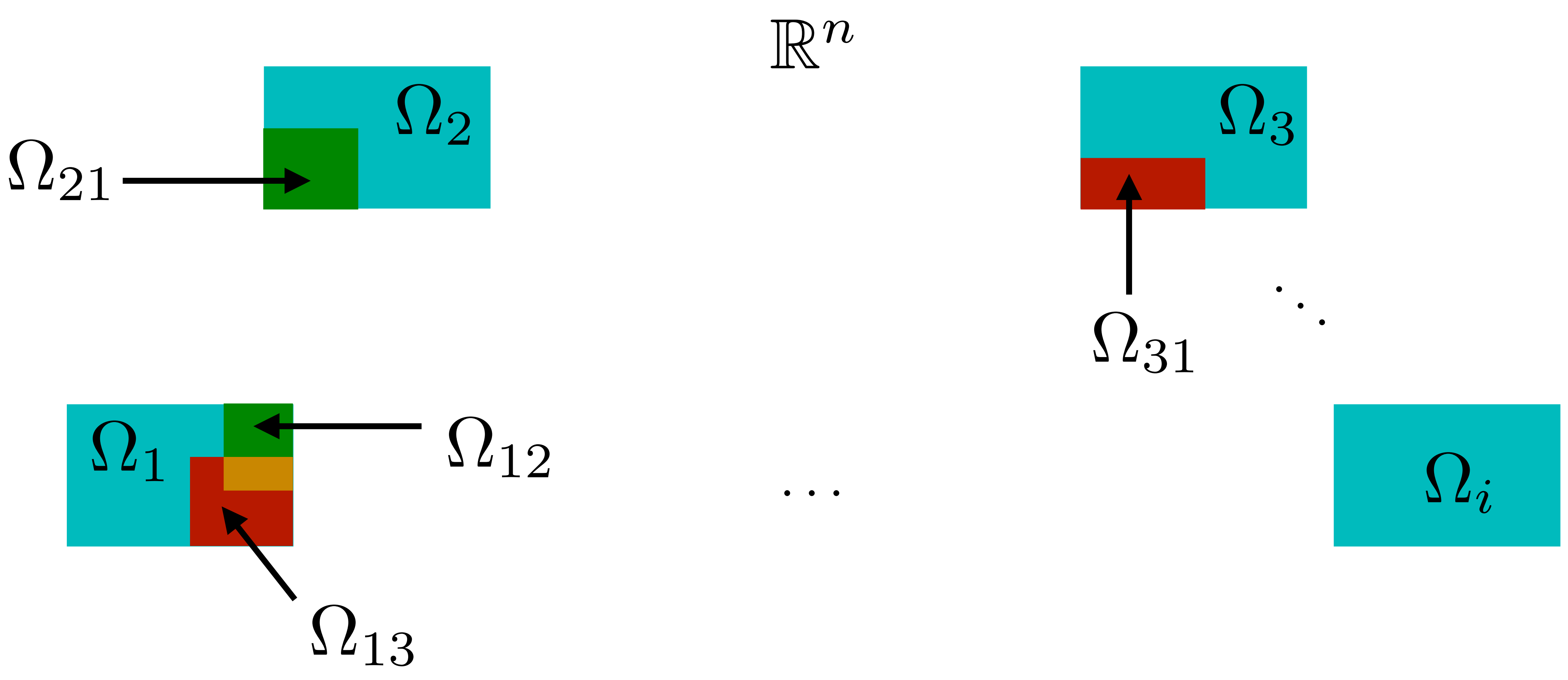

Sets of Gluing Data

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(3) If we let

$$K = \{(i,j) \in I \times I \mid \Omega_{ij} \neq \emptyset\},$$

then

$$\varphi_{ji} \colon \Omega_{ij} \longrightarrow \Omega_{ji}$$

is a C^k bijection for every $(i,j) \in K$, called a **transition function** or **gluing function**.

Transition Functions

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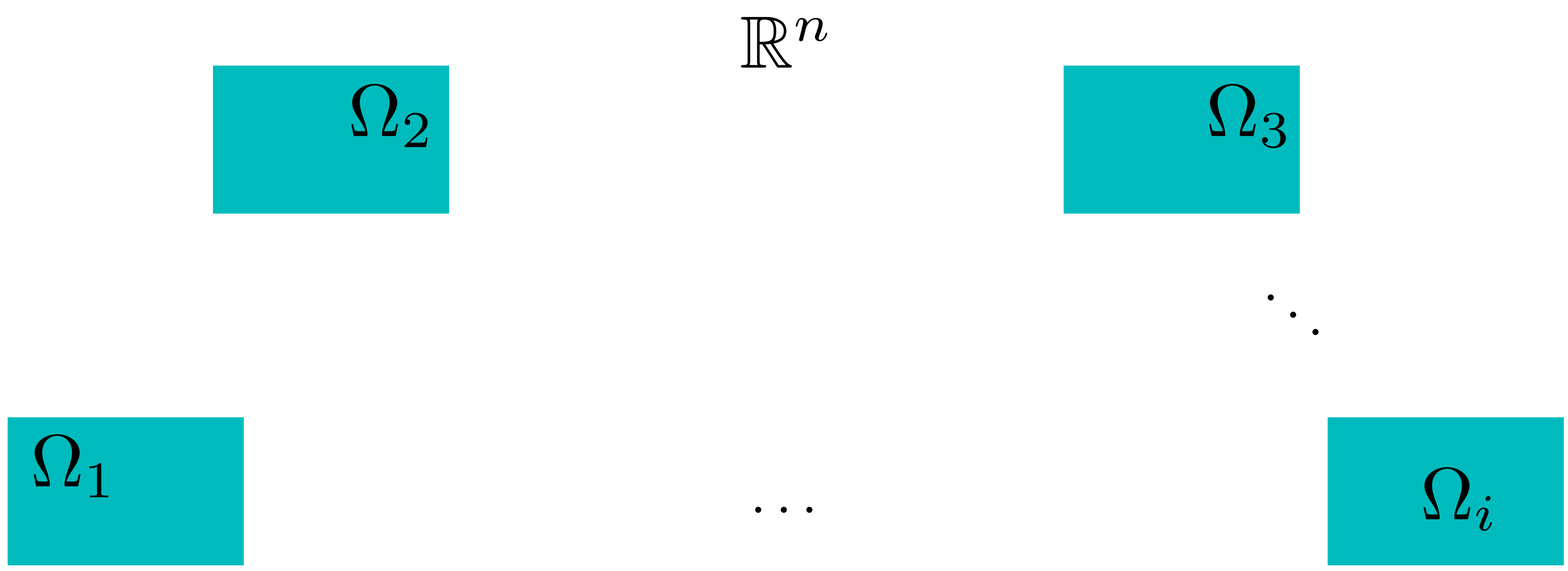

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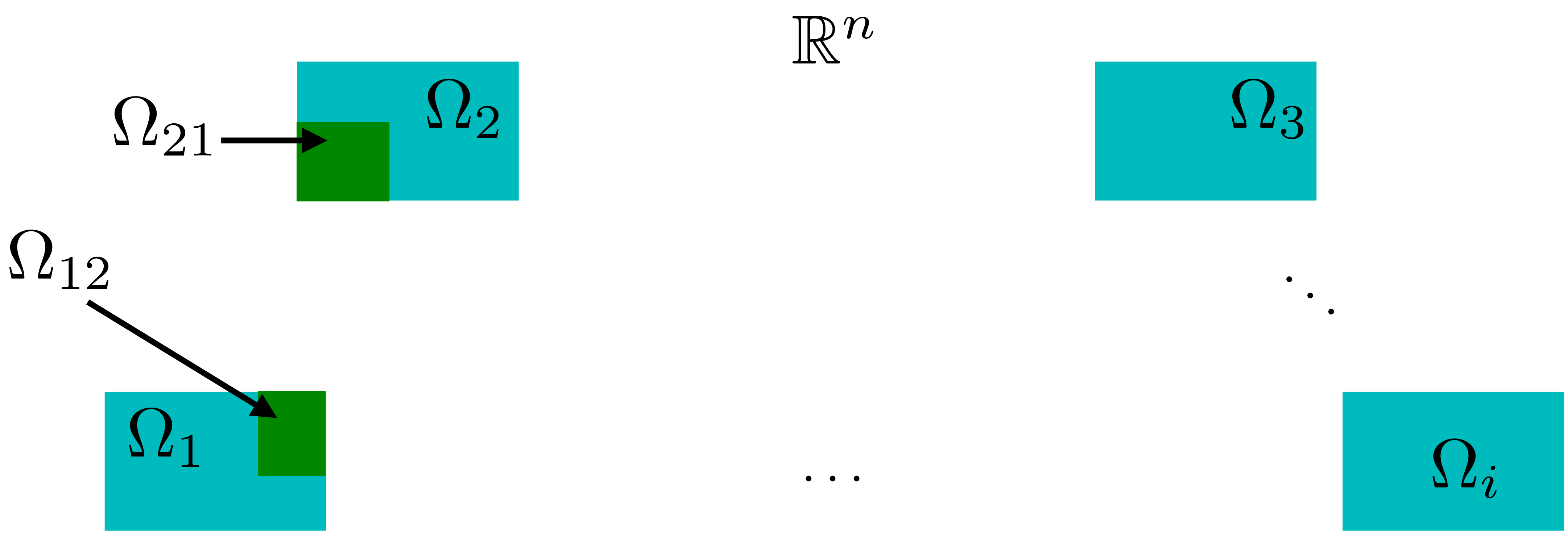

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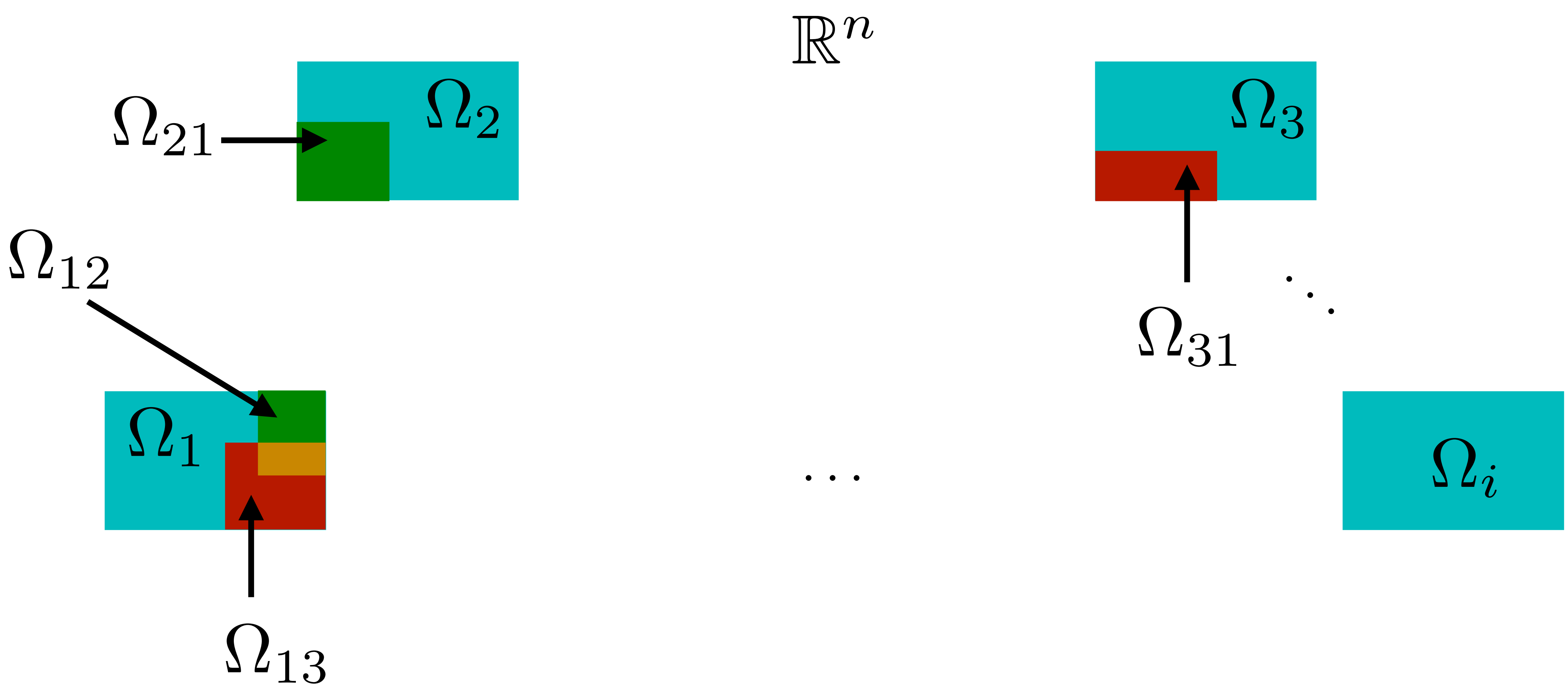

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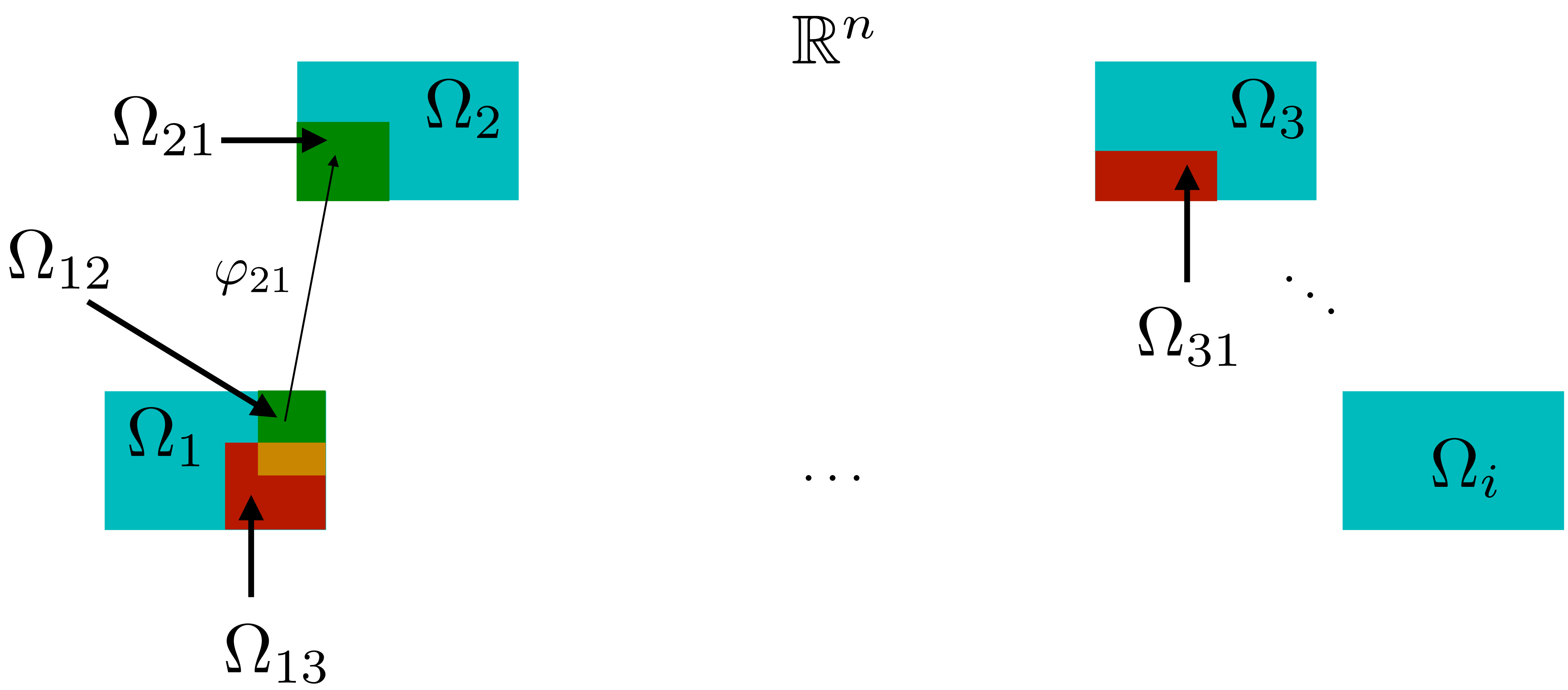

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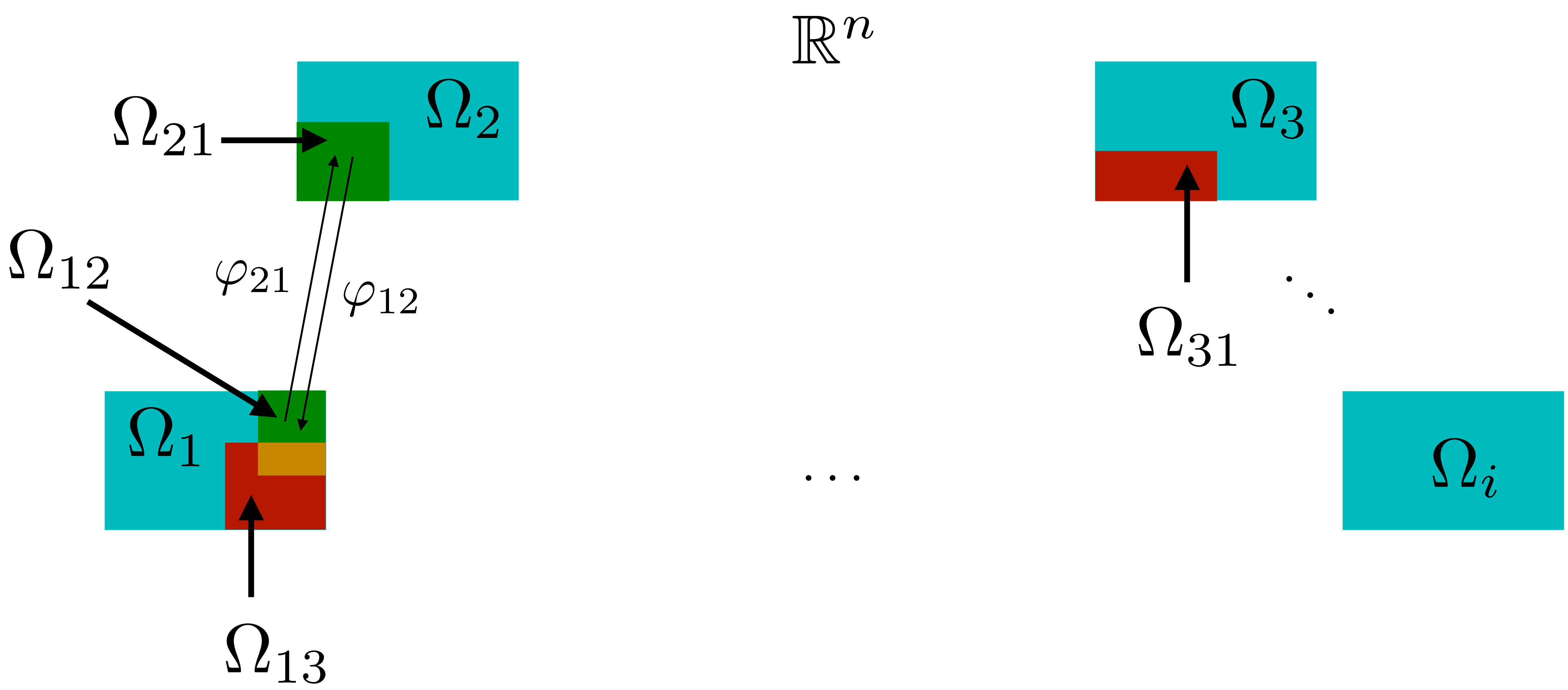

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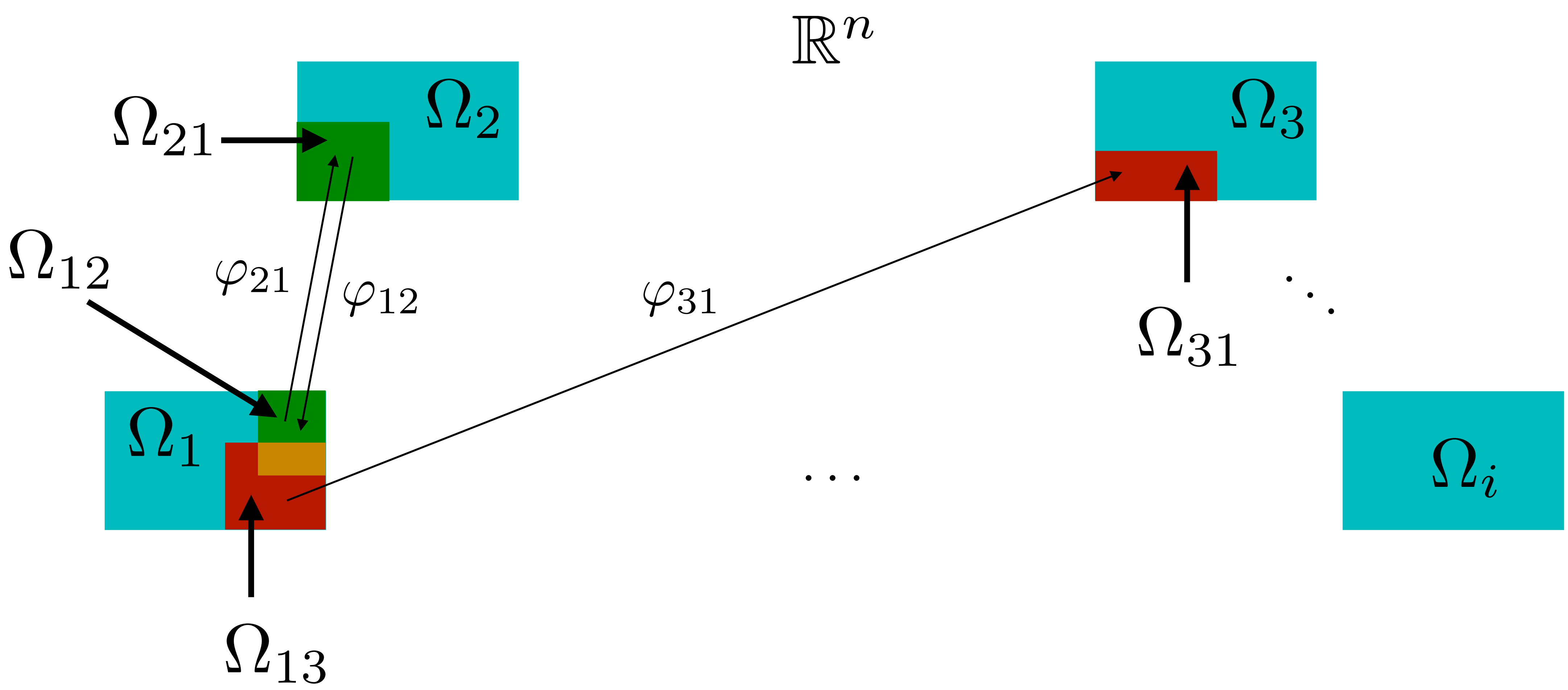

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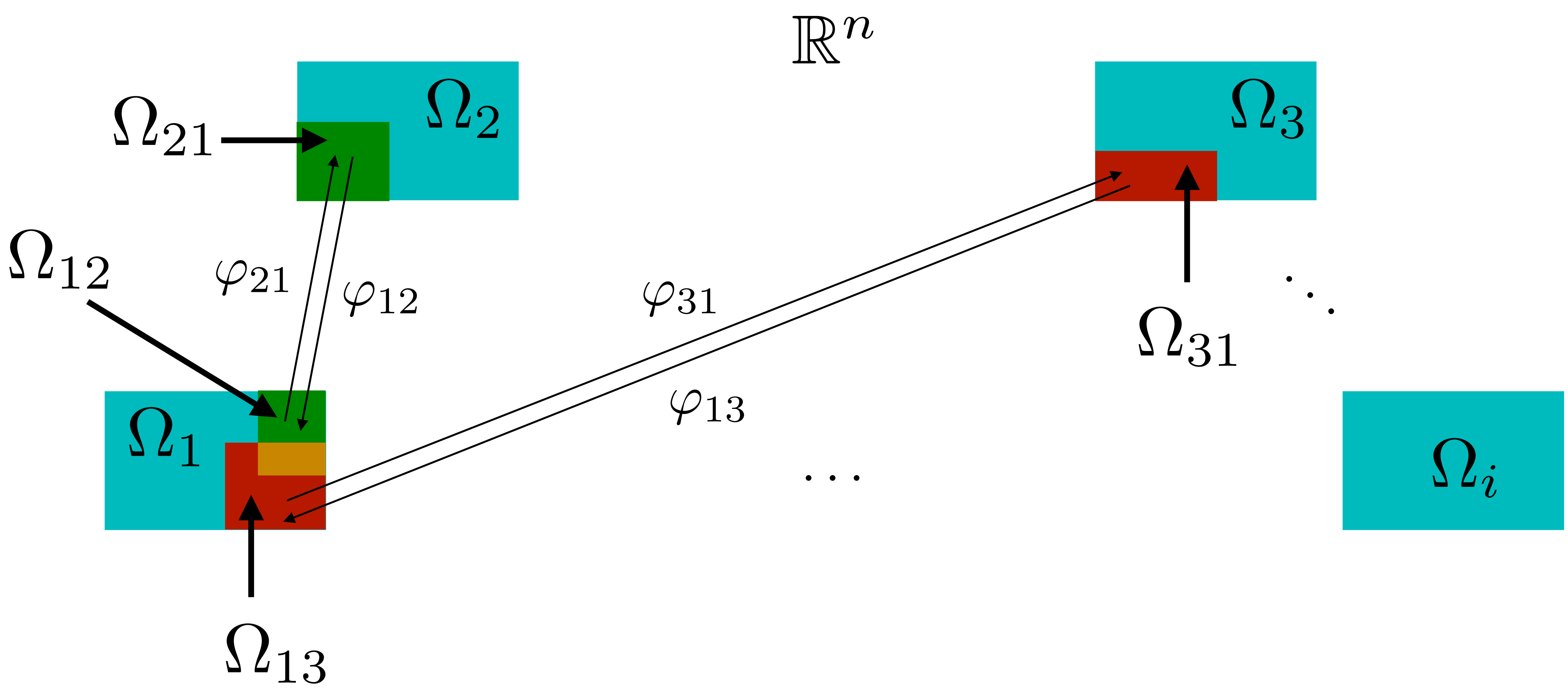

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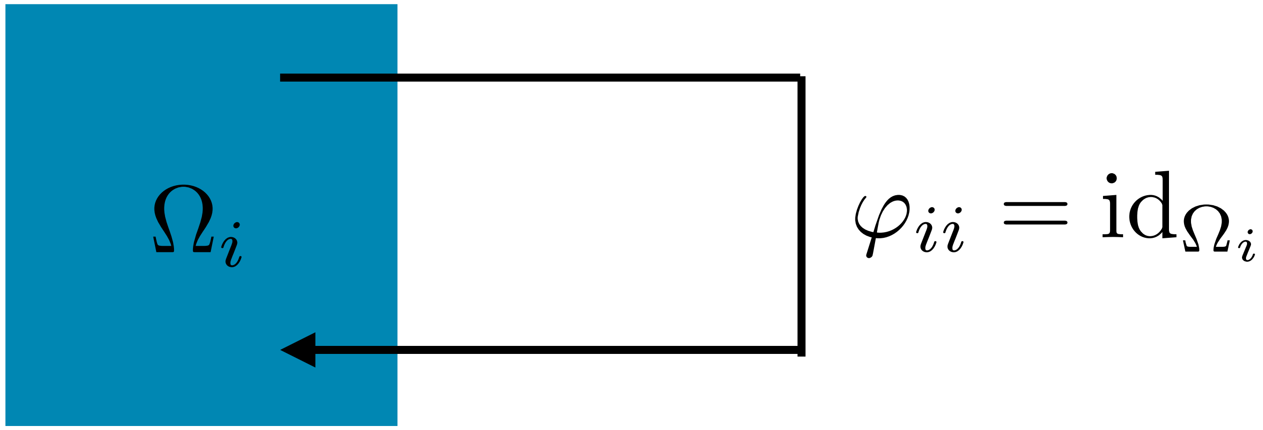
(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$,



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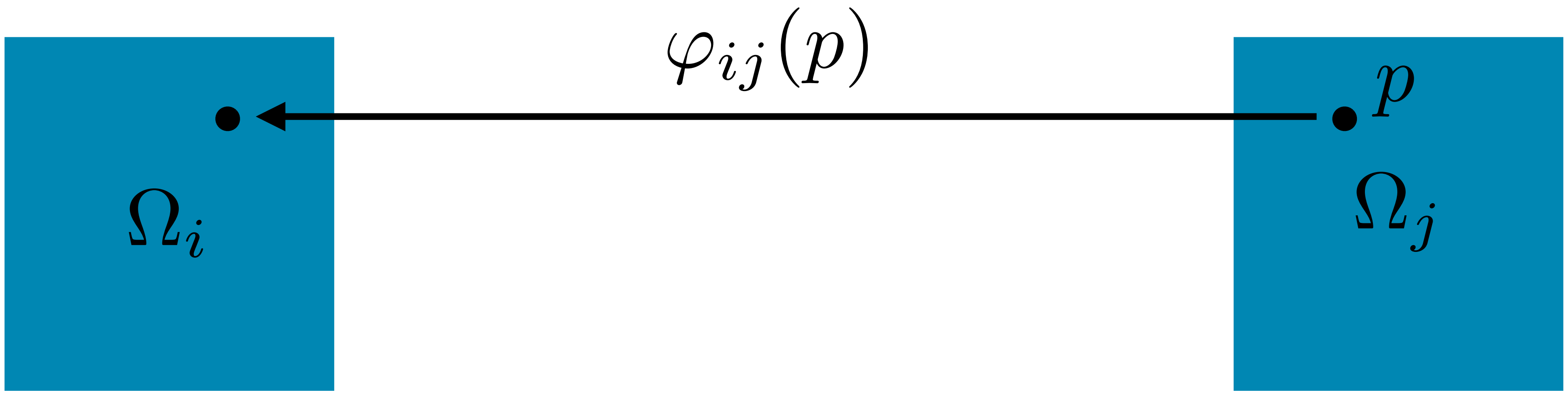

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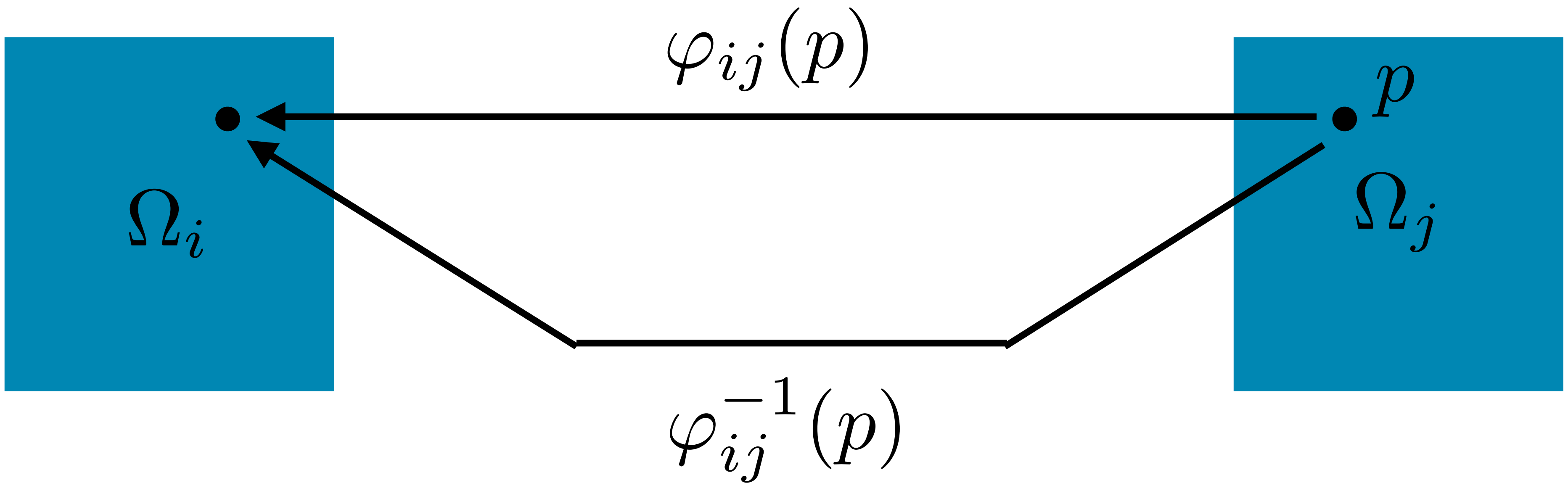

Transition Functions

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(c) for all i, j, and k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$ then $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$ and $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)$, for all $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$.

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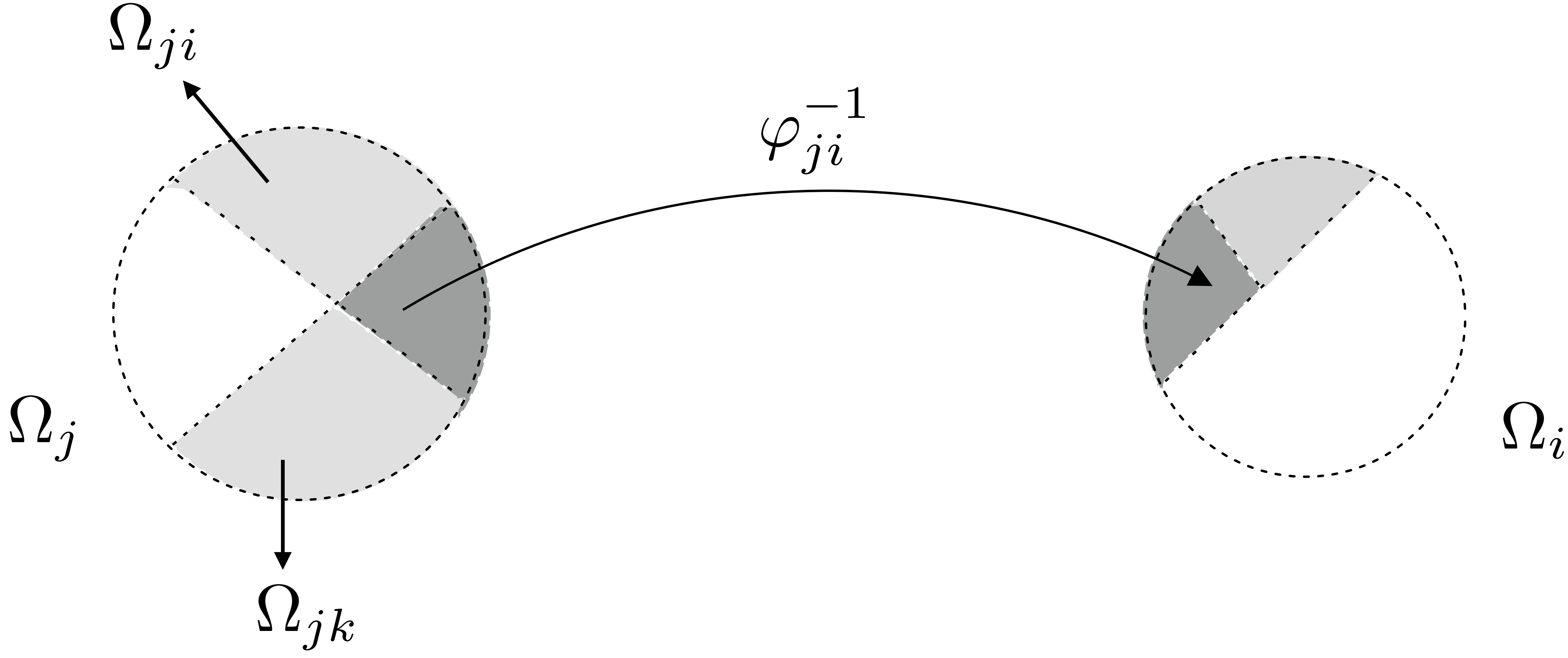

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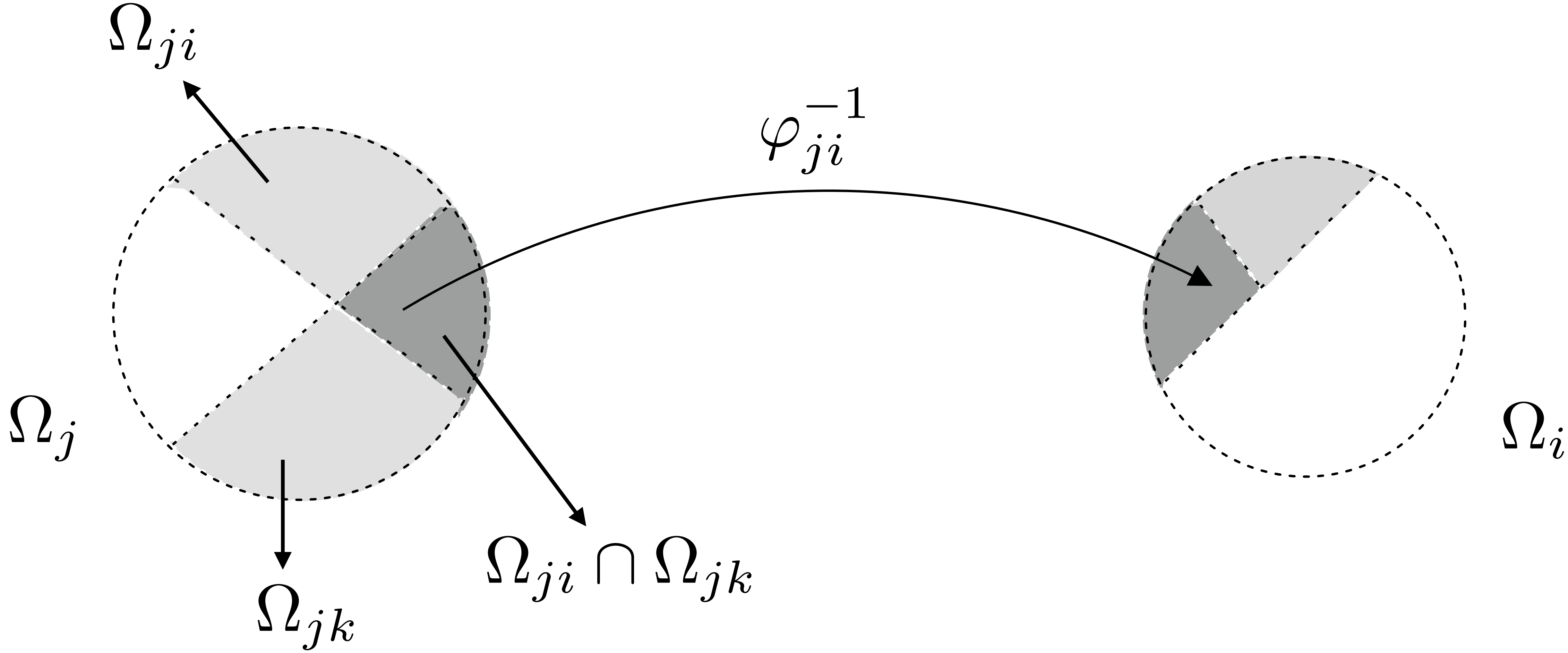

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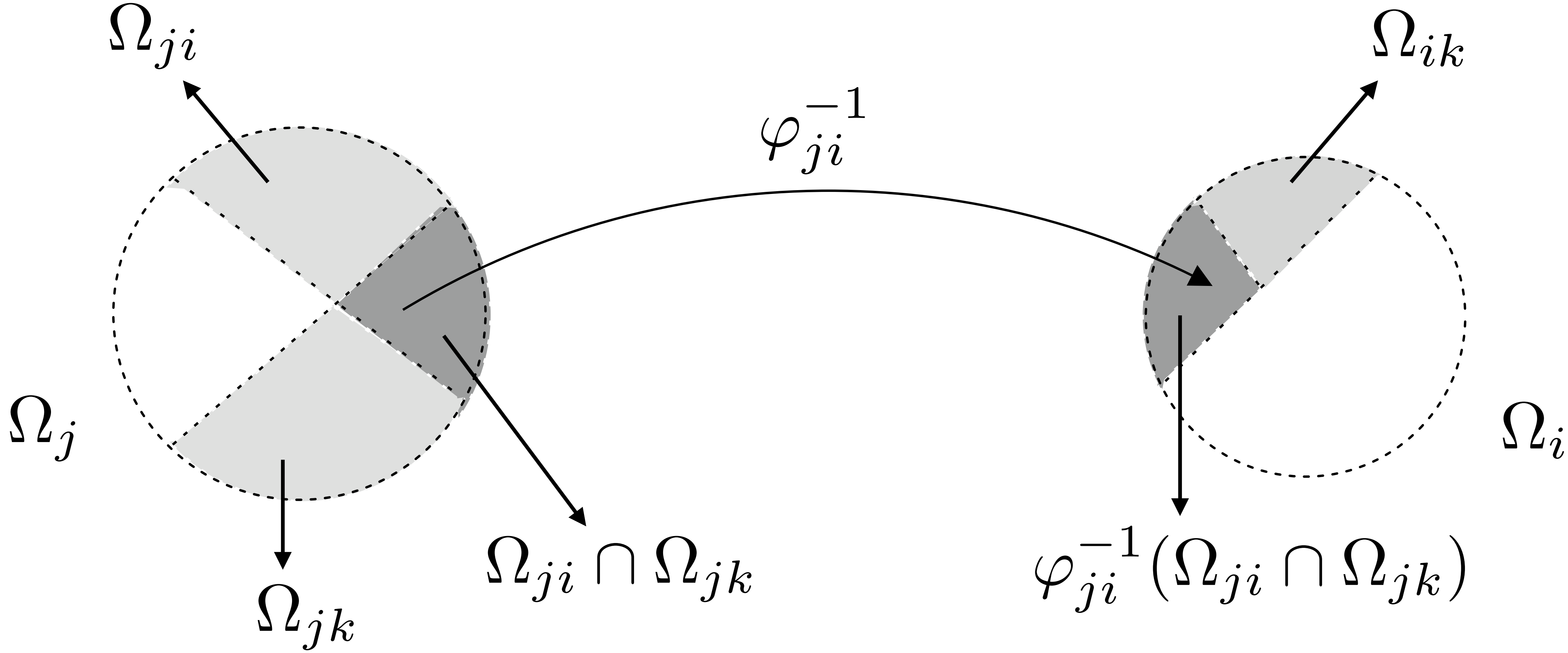

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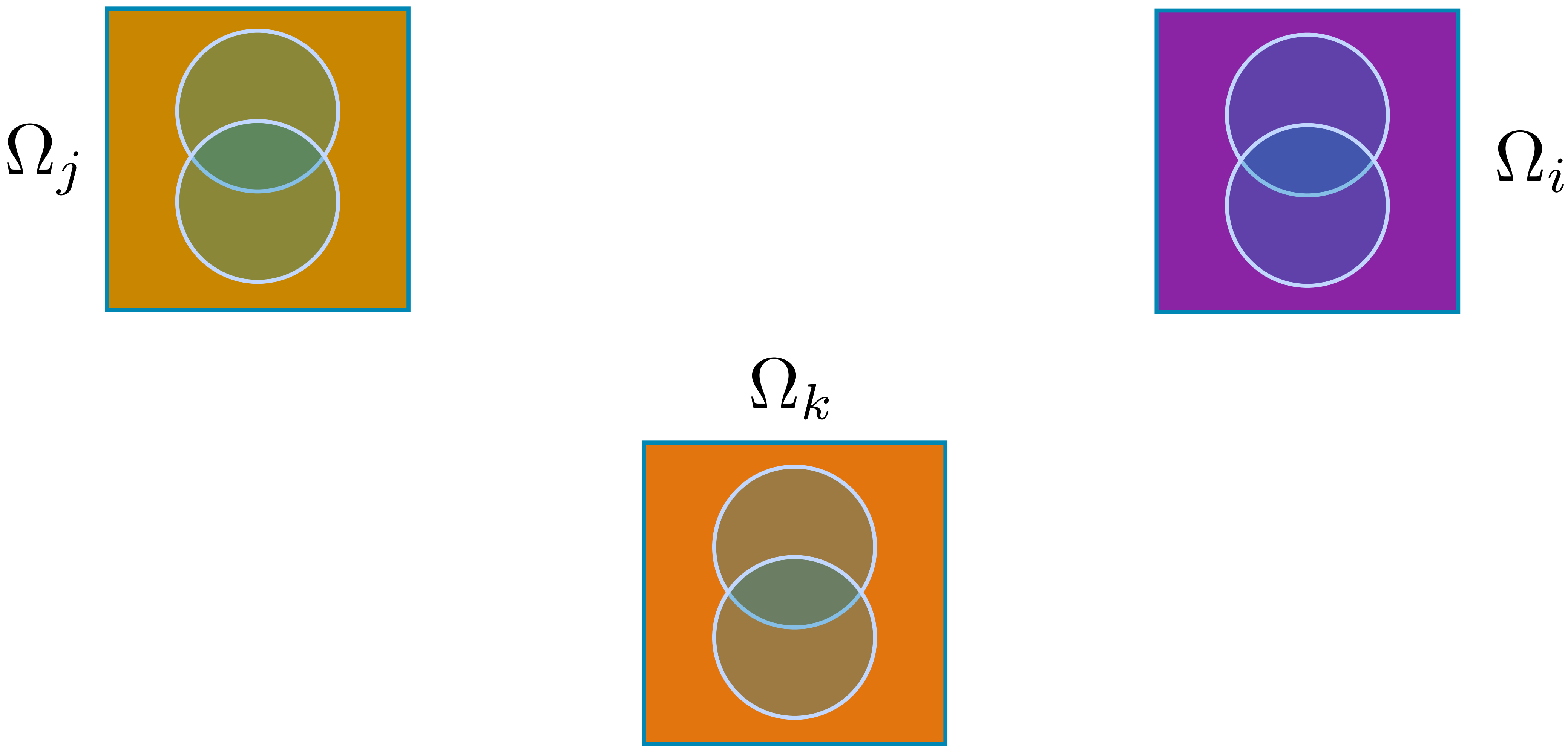

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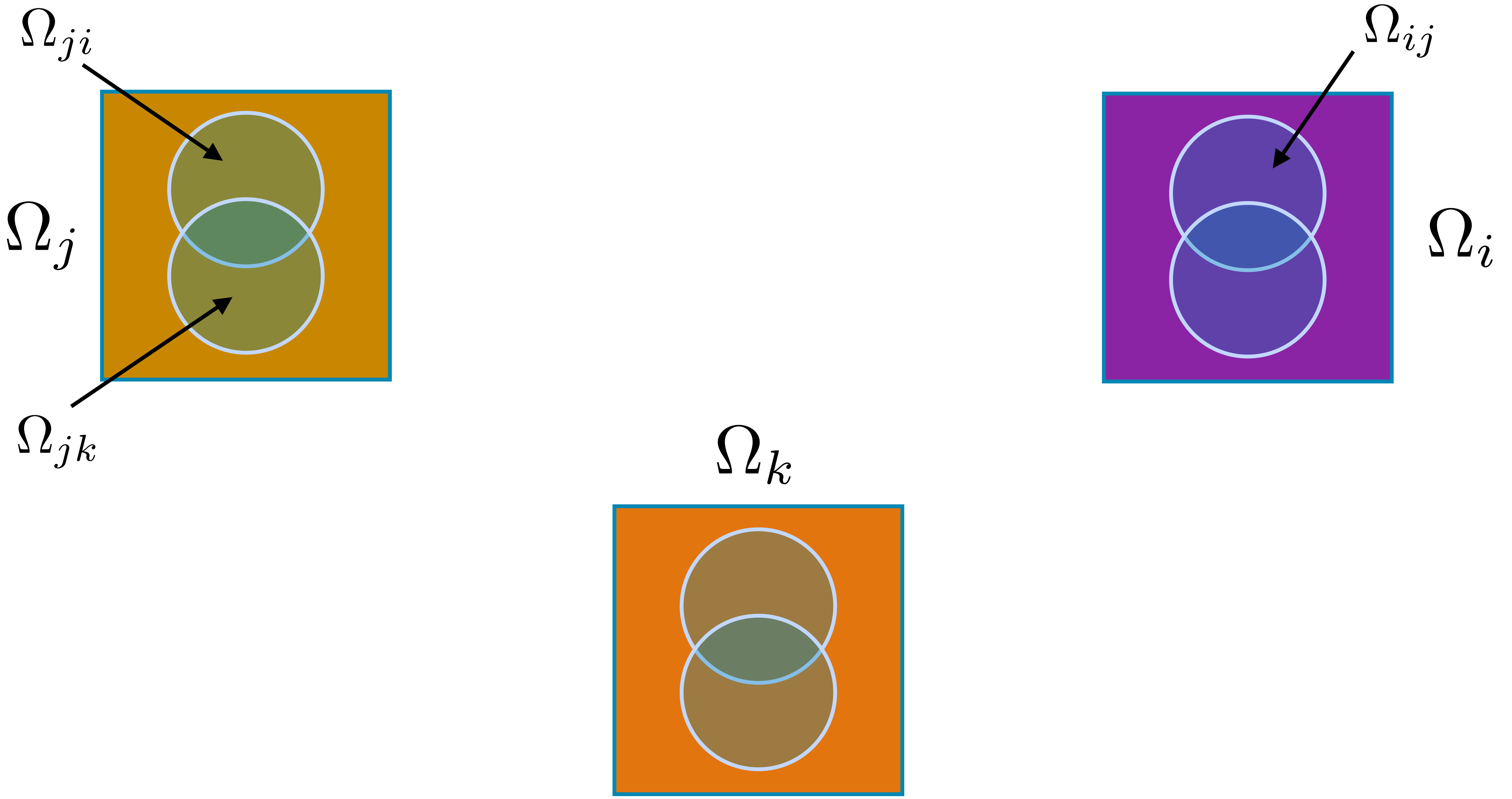

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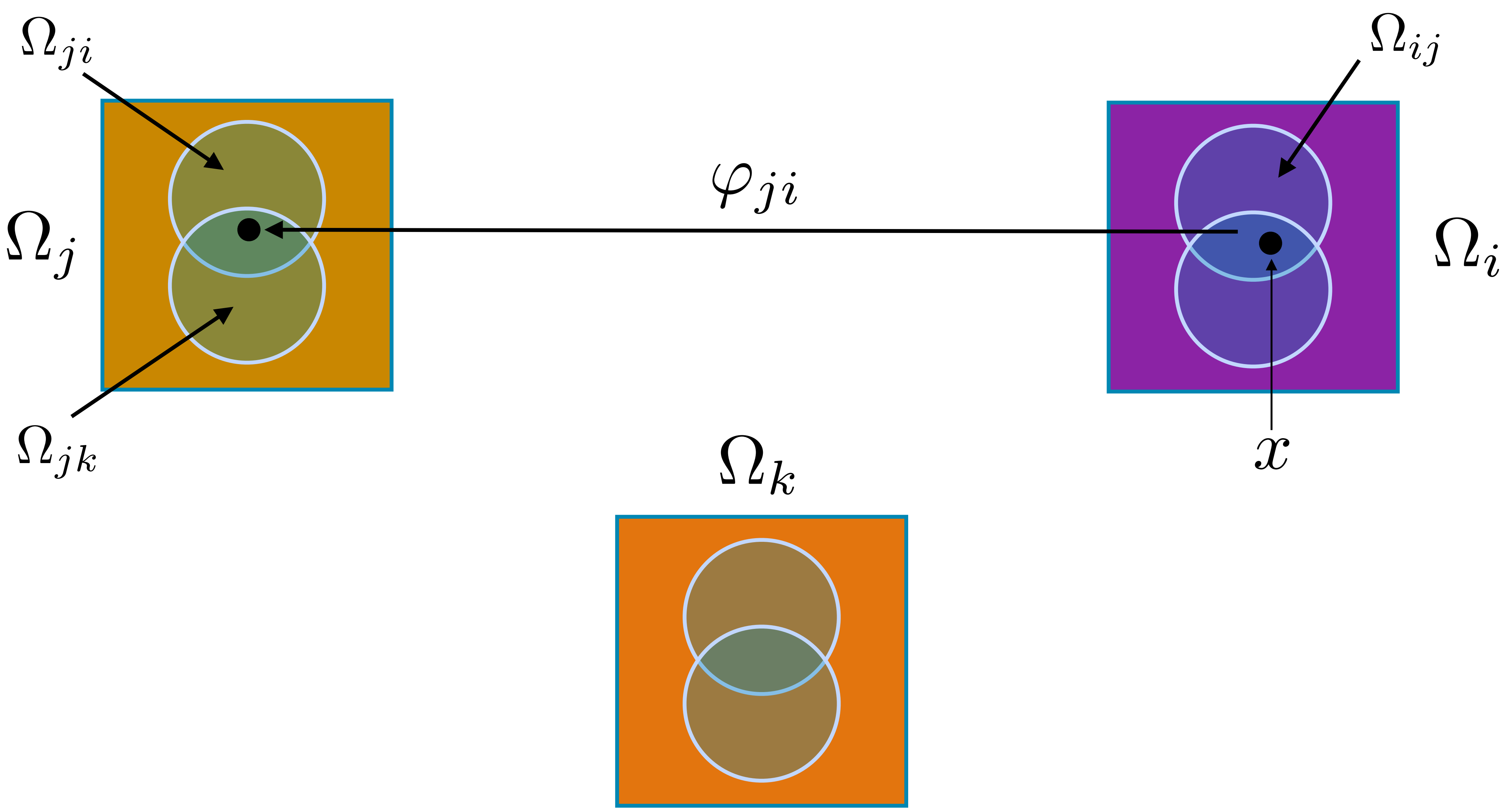

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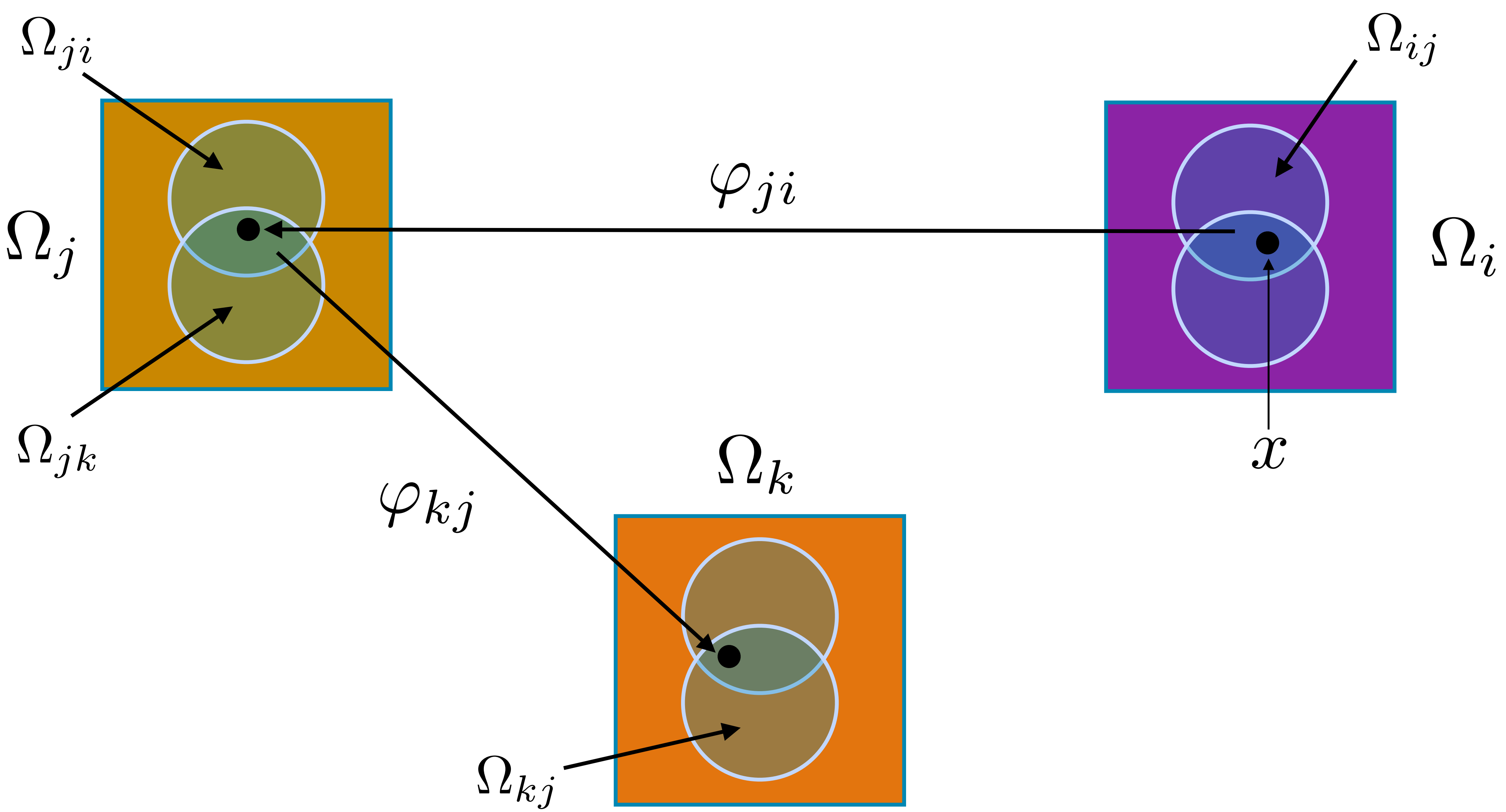

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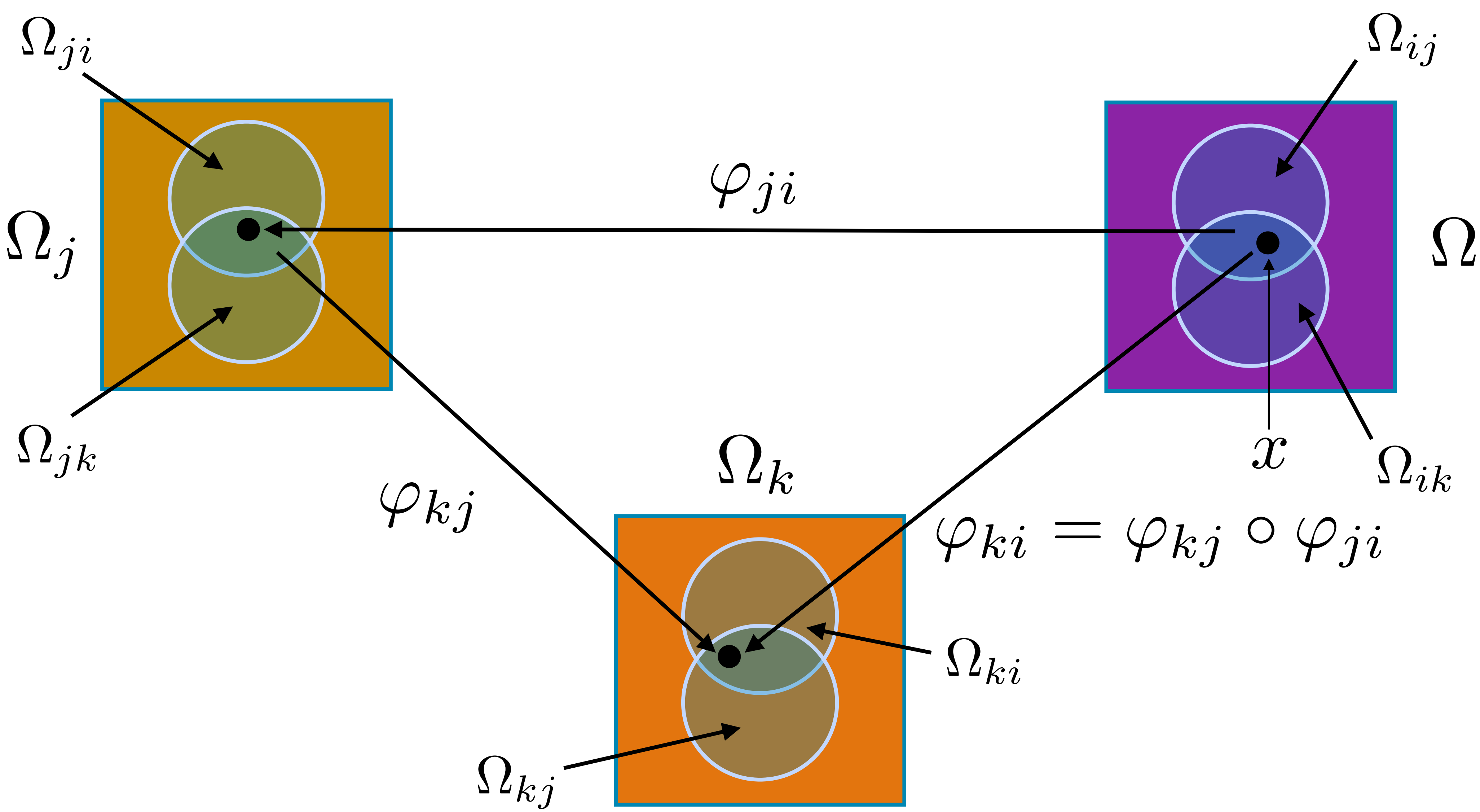

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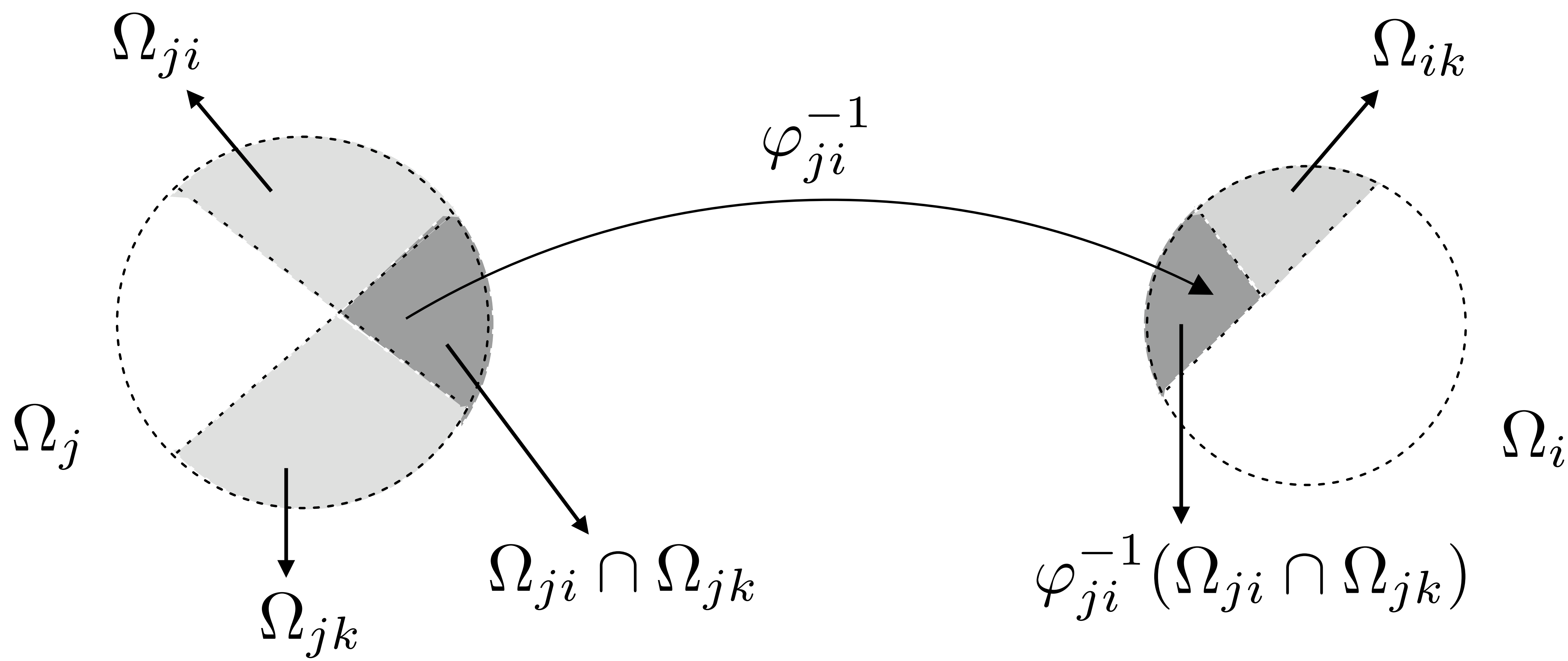

Parametric Pseudo-Manifolds

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- Indeed, such a manifold is built by a **quotient construction**.

- We form the disjoint union of the Ω_i and we identify Ω_{ij} with Ω_{ji} using φ_{ji}, an equivalence relation, $\sim$. We form the quotient

$$M_{\mathcal{G}} = \left(\coprod_i \Omega_i \right) / \sim, \ .$$

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

Theorem 1 [Gallier, Siqueira, and Xu, 2008]

For every set of gluing data,

$$\mathcal{G} = \big((\Omega_i)_{i\in I}, (\Omega_{ij})_{(i,j)\in I\times I}, (\varphi_{ji})_{(i,j)\in K\times K}\big) \ ,$$

there is an n-dimensional C^k manifold, $M_{\mathcal{G}}$, whose transition functions are the φ_{ji}'s.

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

REMARK:

A condition on the gluing data is needed to make sure that $M_{\mathcal{G}}$ is Hausdorff. Since it is quite technical, we will not show it here.

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- Our proof is not constructive;
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Let us first formalize our notion of "concreteness".

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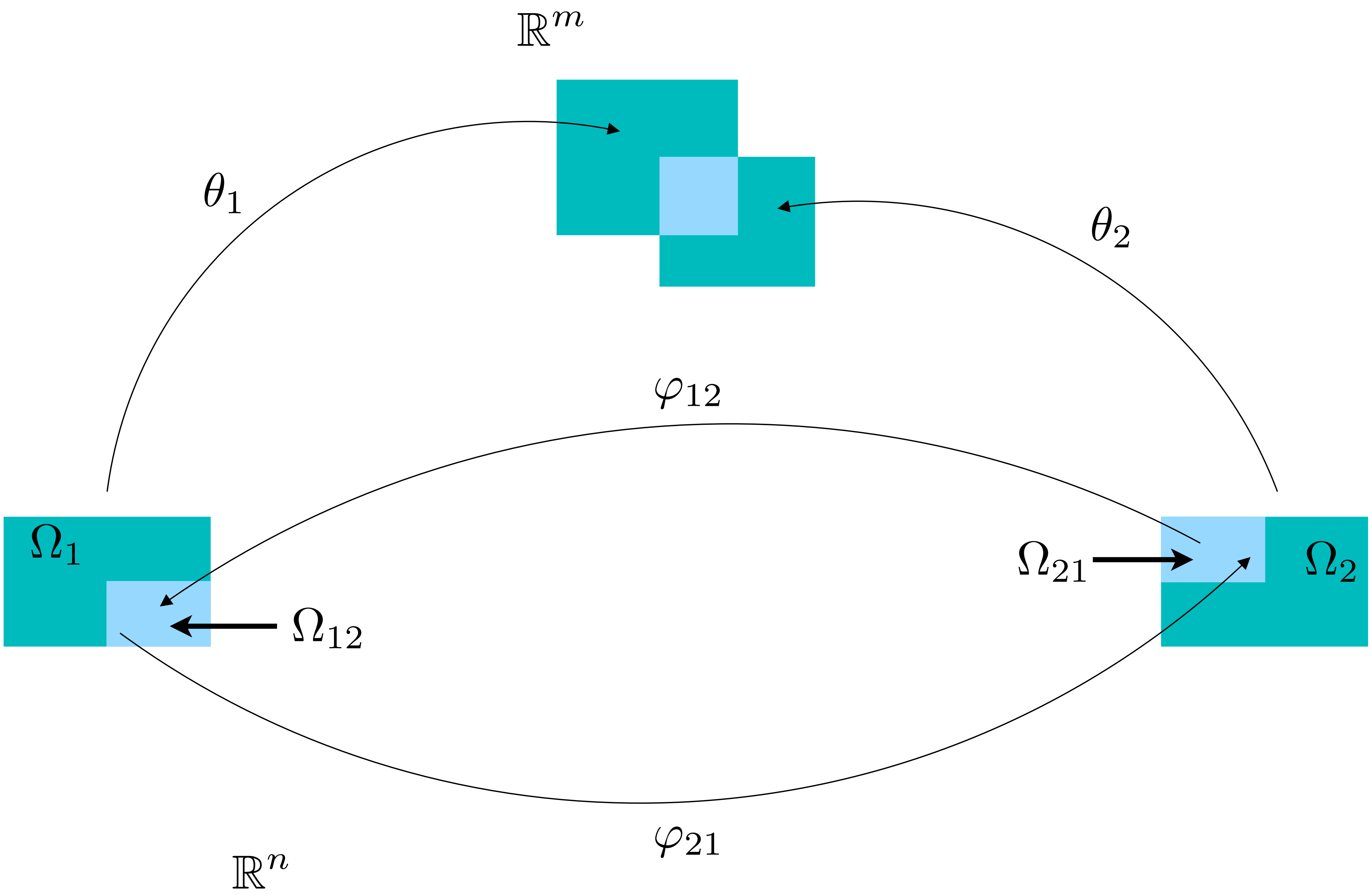
A **parametric** C^k **pseudo-manifold of dimension** n **in** $\mathbb{R}^m$ is a pair,

$$\mathcal{M} = (\mathcal{G}, (\theta_i)_{i \in I}) ,$$

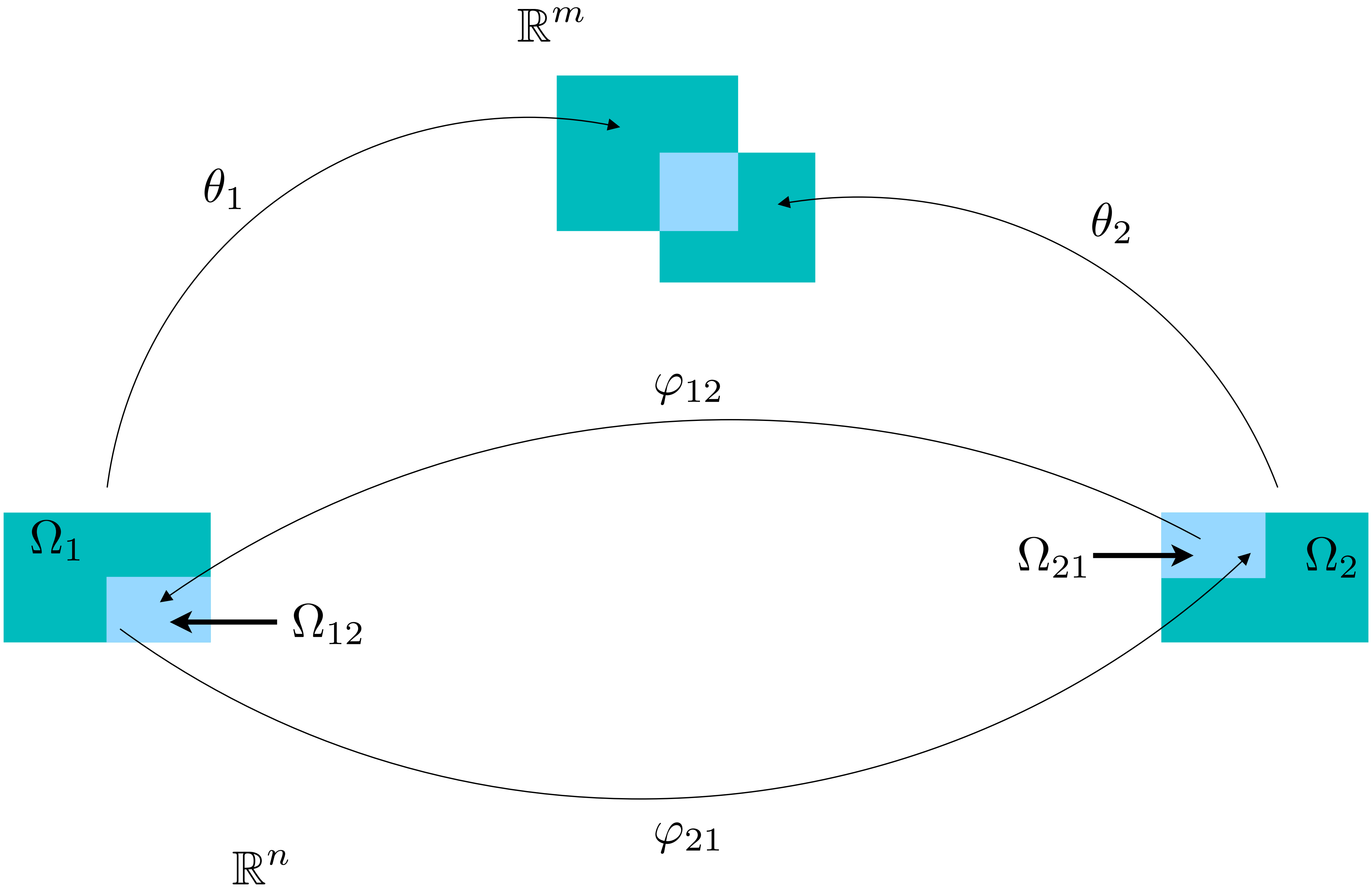
such that $\mathcal{G} = \big((\Omega_i)_{i \in I}, (\Omega_{ij})_{(i,j) \in I \times I}, (\varphi_{ij})_{(i,j) \in K \times K}\big)$ is a set of gluing data, for some finite I, and each θ_i is a C^k function, $\theta_i : \Omega_i \rightarrow \mathbb{R}^m$, called a **parametrization** such that the following holds:

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds



Parametric Pseudo-Manifolds



- When $m = 3$ and $n = 2$, we say that $\mathcal{M}$ is a **parametric pseudo-surface**.

Parametric Pseudo-Manifolds

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(C) For all $(i, j) \in K$, we have $\theta_i = \theta_j \circ \varphi_{ji}$.

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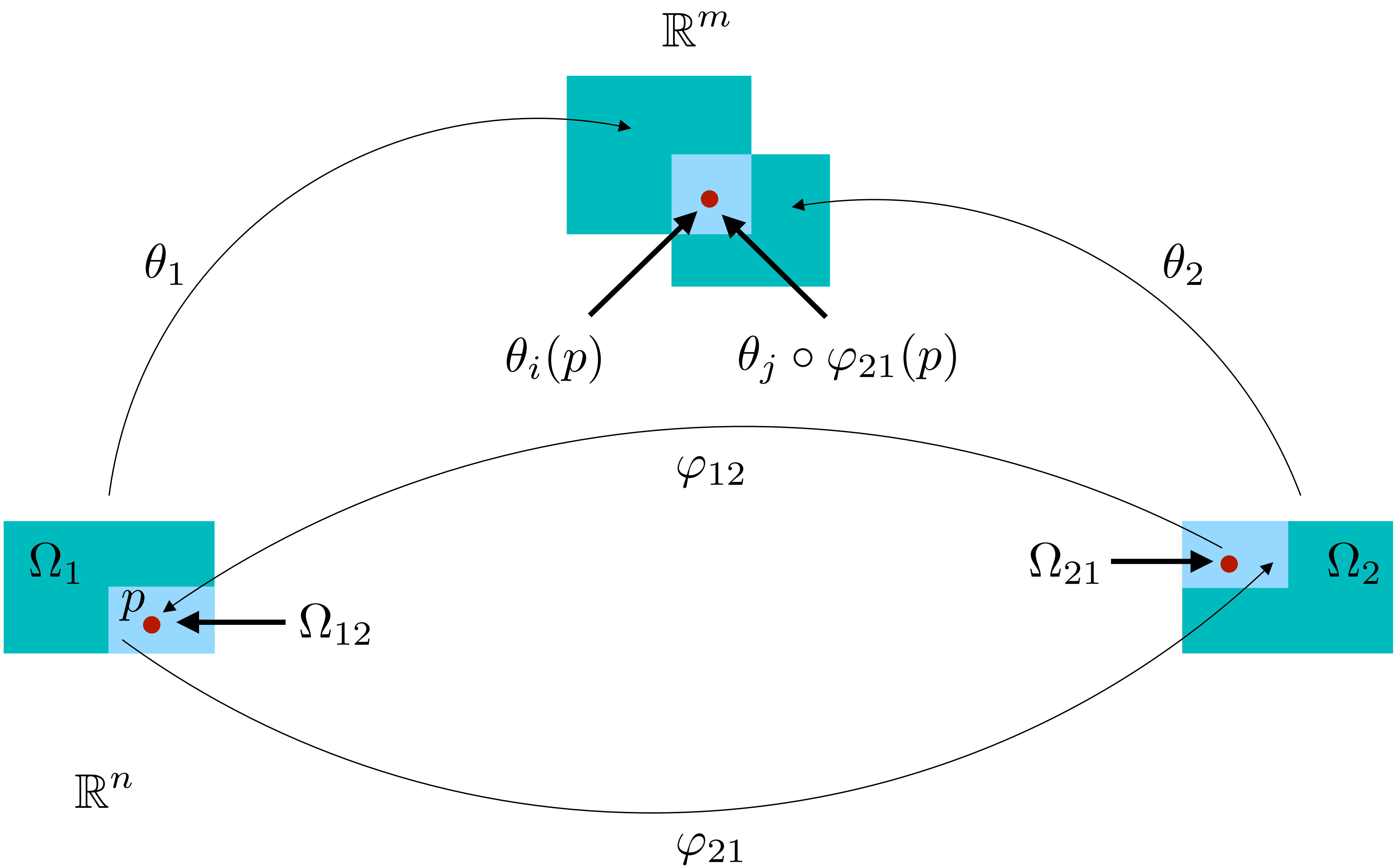

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

- The subset

$$M = \bigcup_{i \in I} \theta_i(\Omega_i)$$

of $\mathbb{R}^m$ is called the **image** of the parametric pseudo-manifold.

Parametric Pseudo-Manifolds

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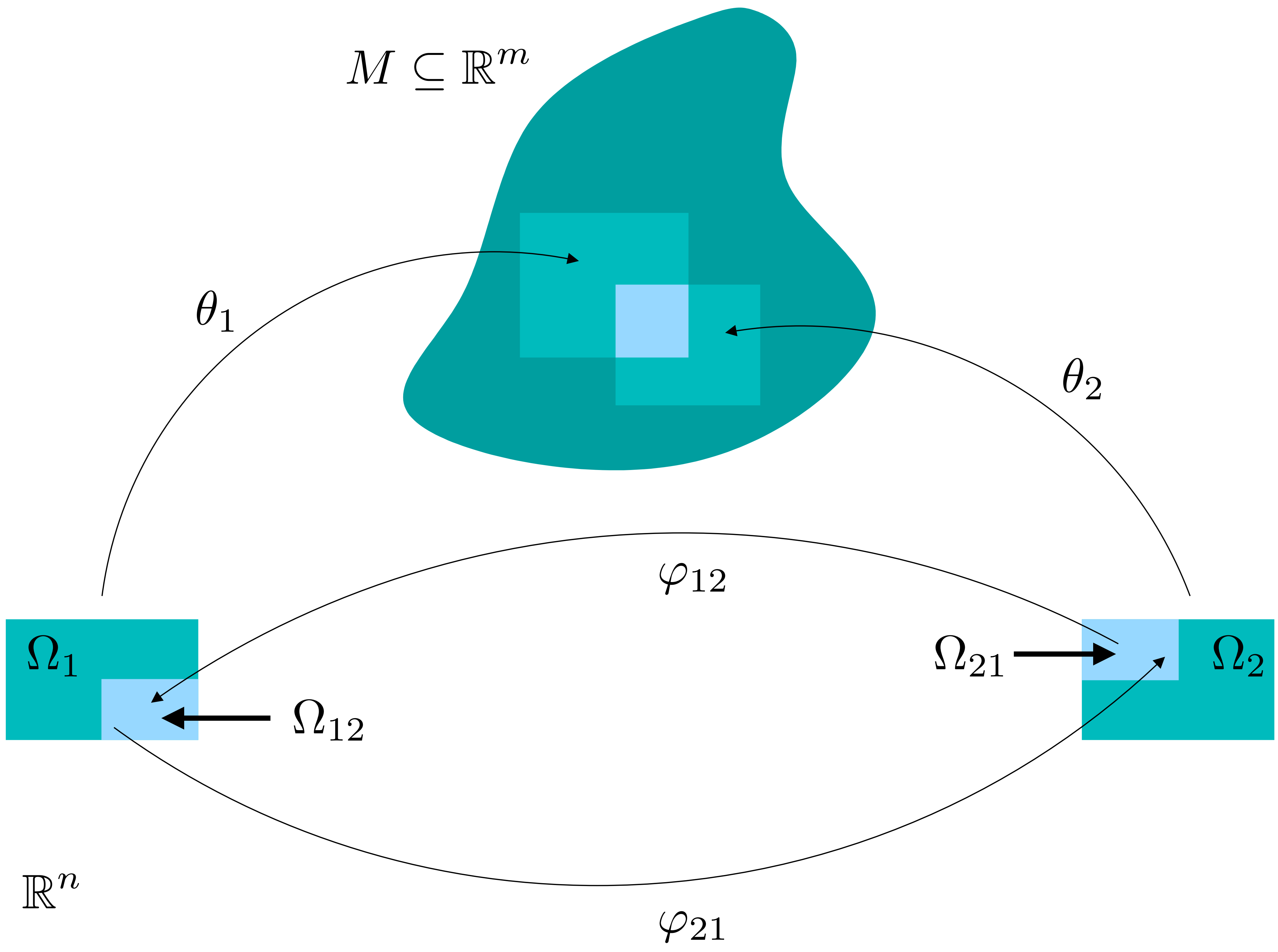

Parametric Pseudo-Manifolds

Parametric Pseudo-Manifolds

REMARK:

There is a (unique) surjective map:

$$\Theta : M_{\mathcal{G}} \longrightarrow M\,.$$

Parametric Pseudo-Manifolds

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(C'') For all $(i, j) \notin K$,

$$\theta_i(\Omega_i) \cap \theta_j(\Omega_j) = \emptyset.$$

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- In the following lecture, we describe a new constructive approach to define a set of gluing data from a triangle mesh.

Conclusions

- We can *build* a parametric pseudo-manifold (PPM) from a set of gluing data and, *under certain conditions*, the image of a PPM can be given the structure of a manifold.
- In the following lecture, we describe a new constructive approach to define a set of gluing data from a triangle mesh.
- We also describe how to build a parametric C^∞ pseudo-surface from the set of gluing data. The image of this parametric pseudo-surface approximates the vertices of the mesh.

Fitting Surfaces to Polygonal Meshes (Part I)

Marcelo Siqueira
UFMS

Outline

- The Surface Fitting Problem
- Building a Set of Gluing Data

The Surface Fitting Problem

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Given a mesh S_T in $\mathbb{R}^3$, a positive integer k, and a positive real number ϵ, our goal here is to fit a C^k surface, S, in $\mathbb{R}^3$ to S_T.

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The Manifold-Based Approach:

We solve the fitting problem by defining a C^k parametric pseudo-surface, $\mathcal{M}$, such that S is the image, M, of $\mathcal{M}$ in $\mathbb{R}^3$.

The Surface Fitting Problem

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Key Idea:

The Surface Fitting Problem

<u>**Key Idea**</u>:

- Use S_T to define the set of gluing data, $\mathcal{G}$, of $\mathcal{M}$.

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- Use $|S_T|$ to define the set of parametrizations, $(\theta_i)_{i \in I}$, of $\mathcal{M}$.

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GEOMETRY

Building a Set of Gluing Data

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Building a Set of Gluing Data

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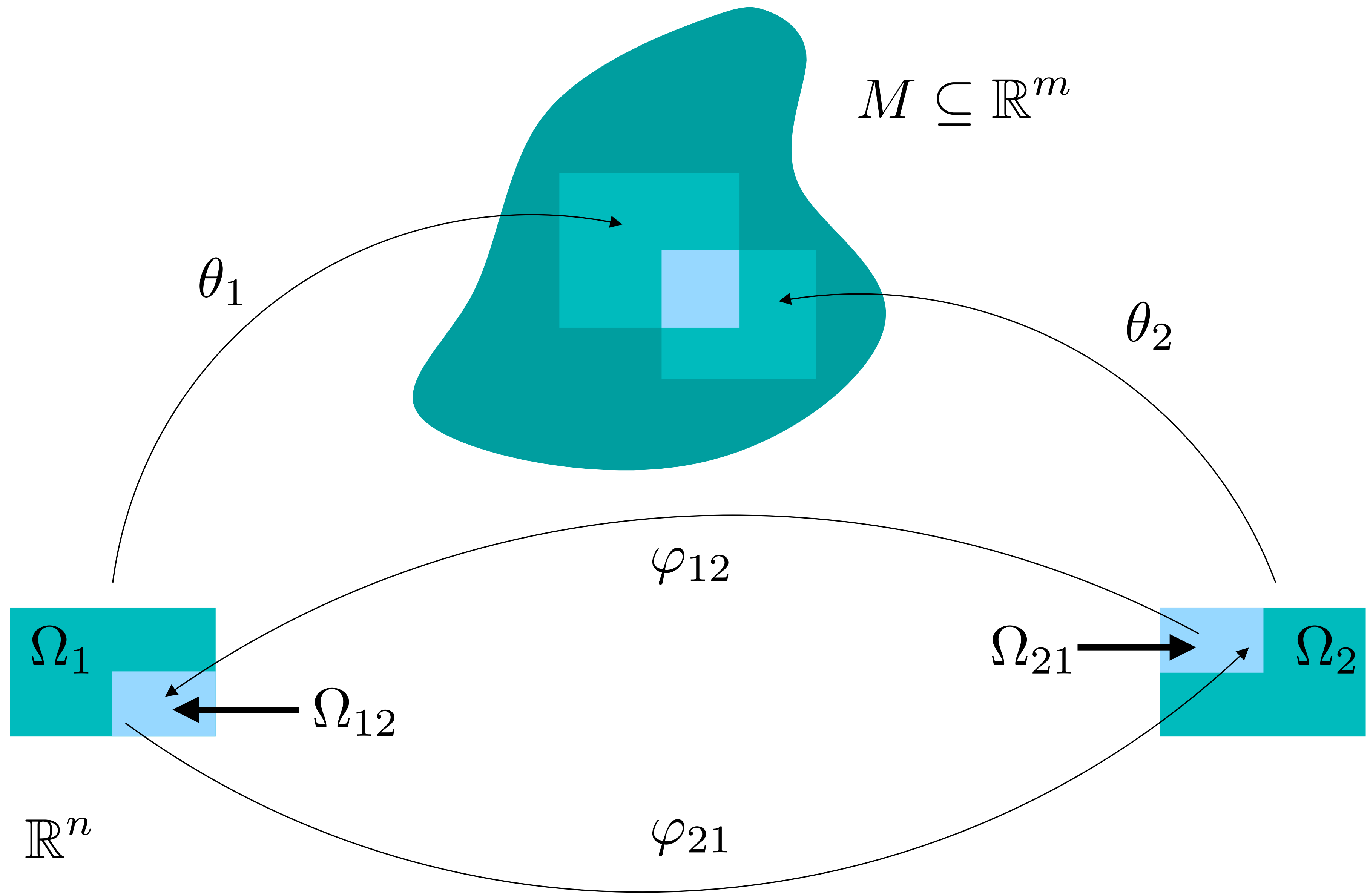
$$\mathcal{G} = \big((\Omega)_{i\in I}, (\Omega_{i,j})_{(i,j)\in I\times I}, (\varphi_{i,j})_{(i,j)\in K\times K}\big)$$

Building a Set of Gluing Data

Building a Set of Gluing Data

The BIG PICTURE



Building a Set of Gluing Data

Building a Set of Gluing Data

p-Domains

Building a Set of Gluing Data

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Assume that S_T is a **triangle** mesh (i.e., a simplicial surface).

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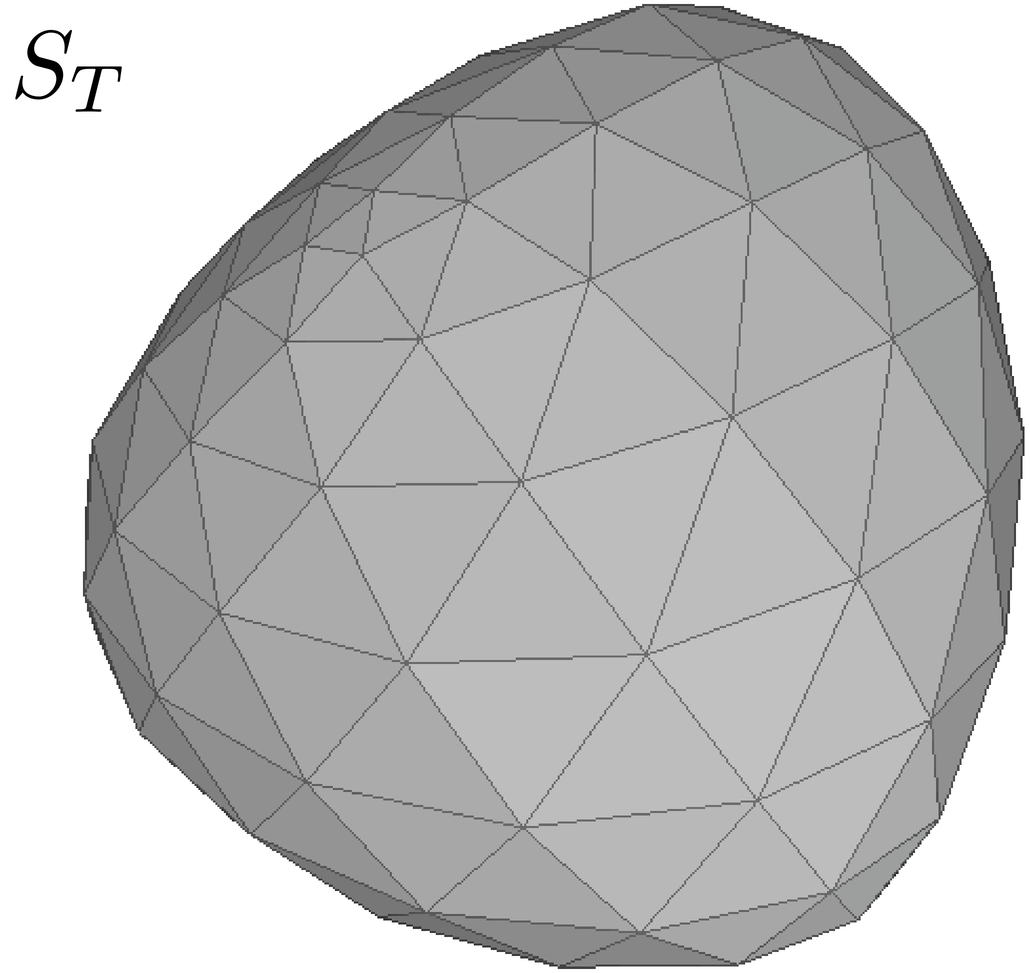

Building a Set of Gluing Data

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$$I = \{(\sigma, v) \mid \sigma \text{ is a triangle of } S_T \text{ and } v \text{ is a vertex of } \sigma\}.$$

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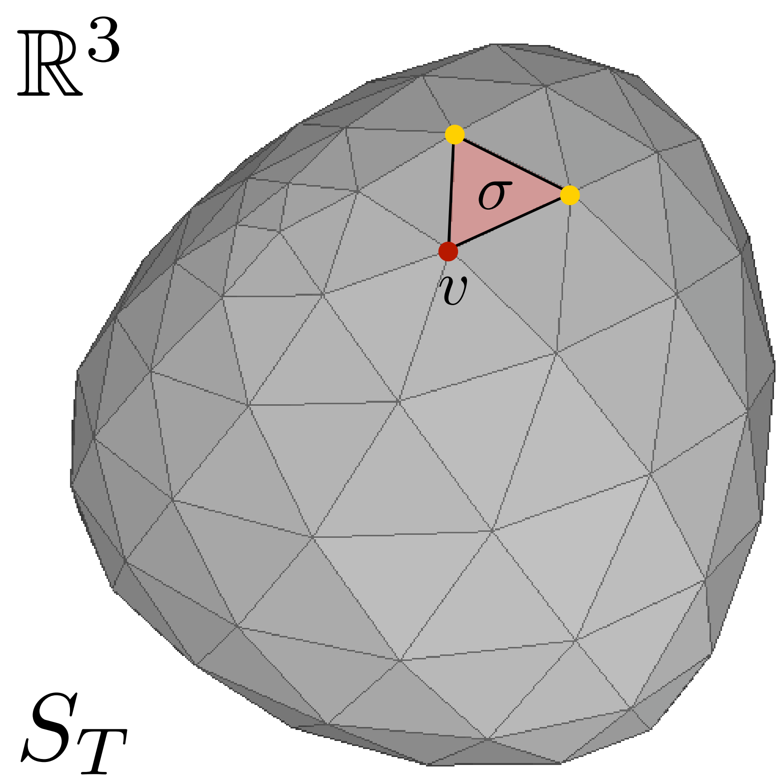
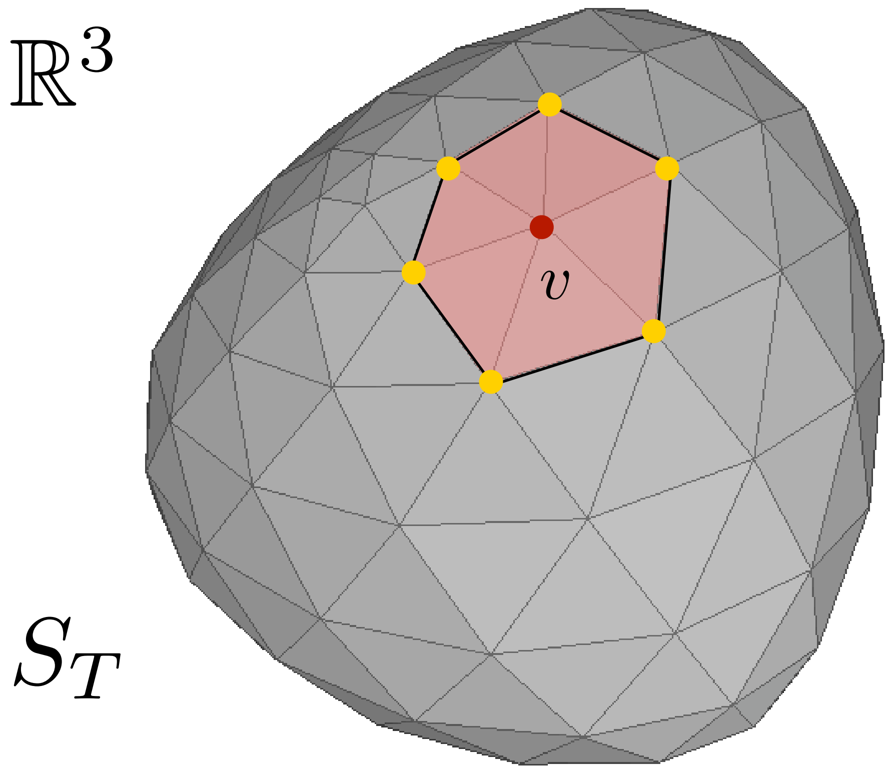


Building a Set of Gluing Data

Building a Set of Gluing Data

For every vertex, v, of S_T, consider its **star**, $st(v, S_T)$:



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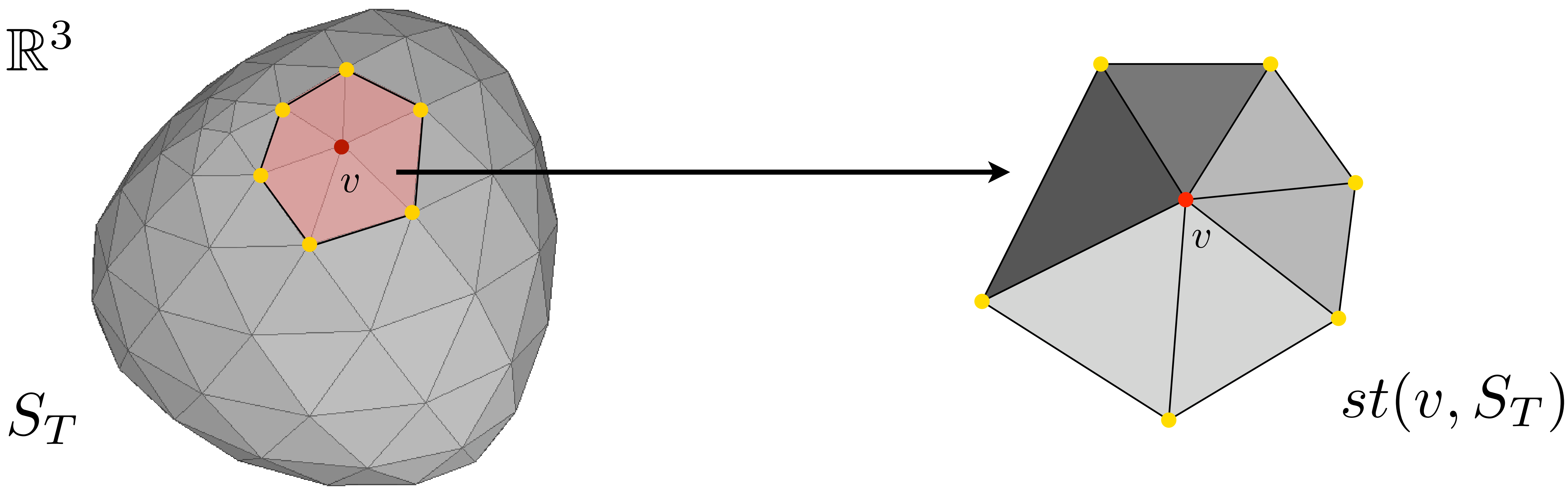
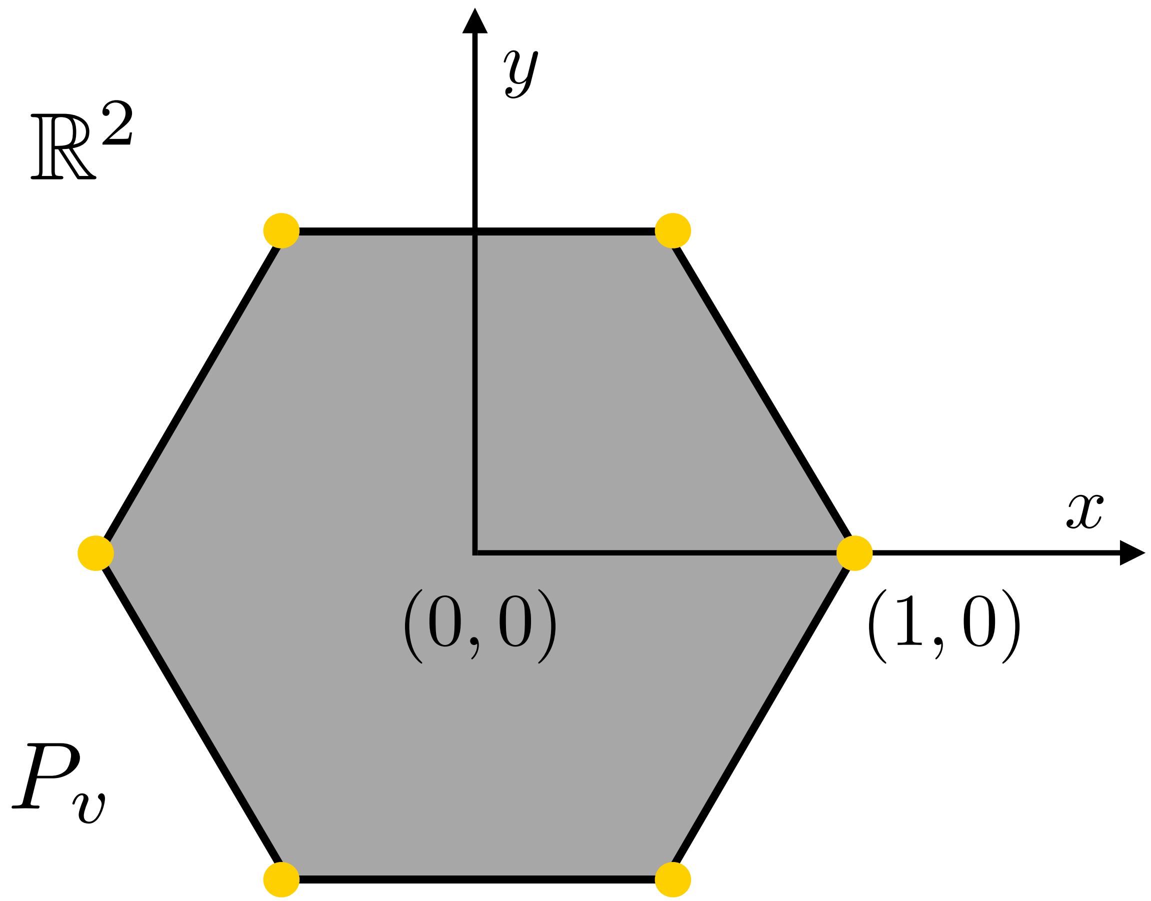


Building a Set of Gluing Data

Building a Set of Gluing Data

Define the **P-polygon, P_v, associated with** v as the m_v-gon inscribed in the circle of radius 1 and centered at the origin in $\mathbb{R}^2$:



m_v is the **degree** of v in S_T.

Building a Set of Gluing Data

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Define the **triangulation,** T_v**, associated with** v by adding straight edges (diagonals) connecting the barycenter of P_v to its vertices:

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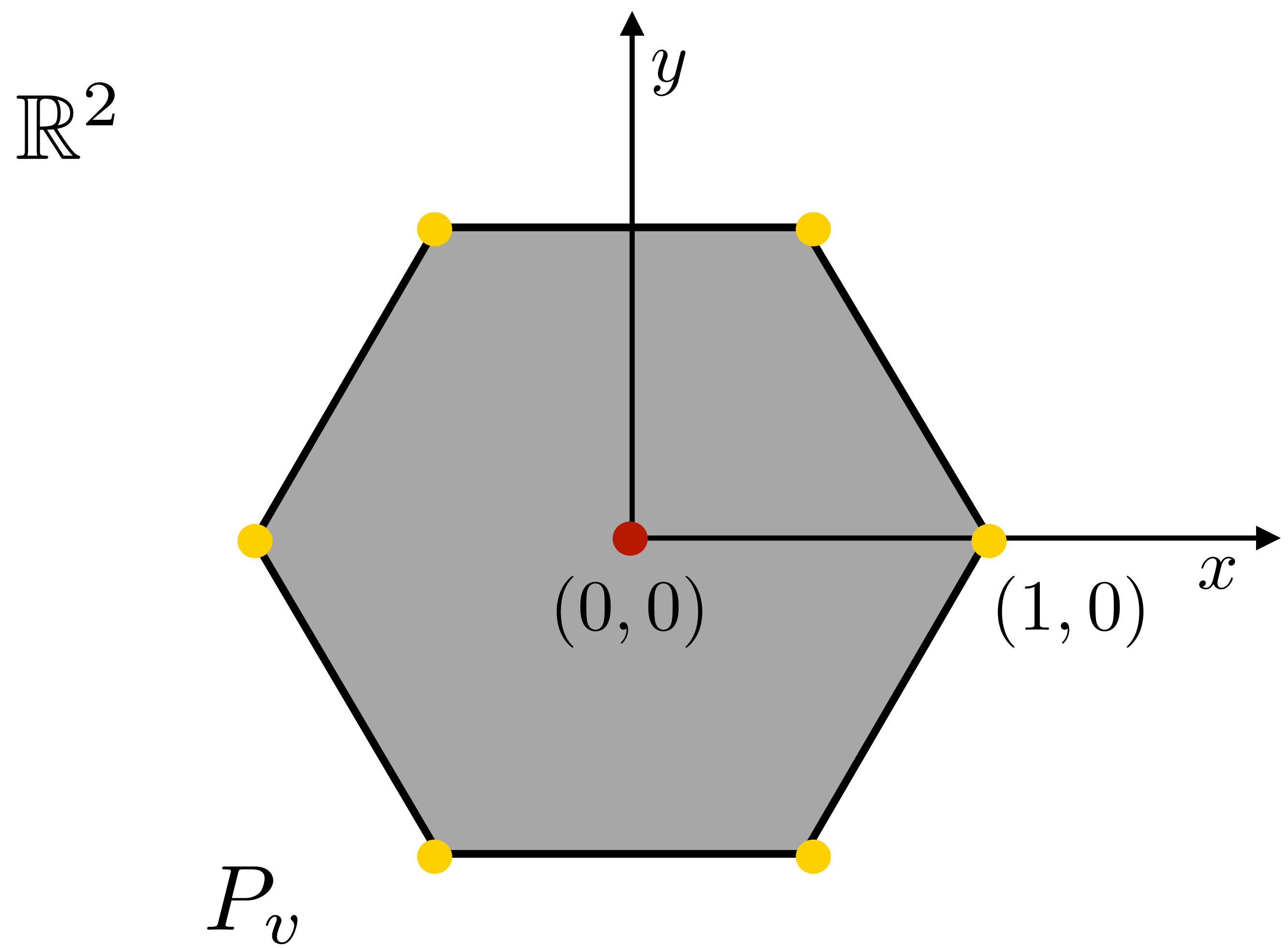

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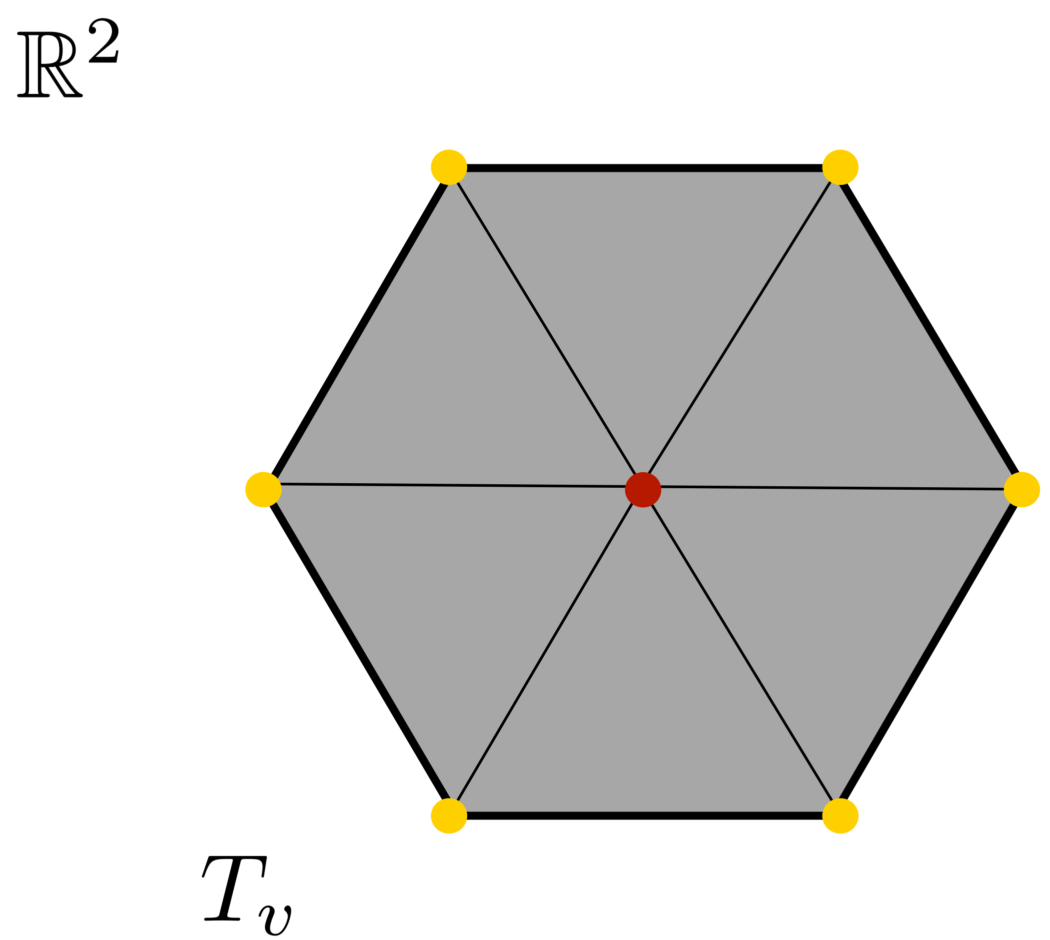

Building a Set of Gluing Data

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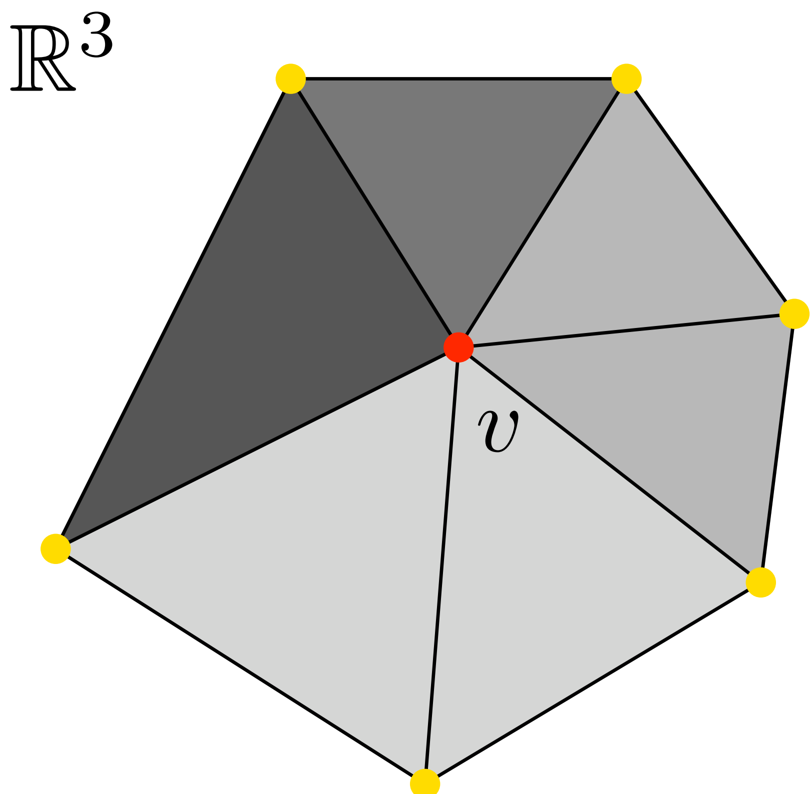
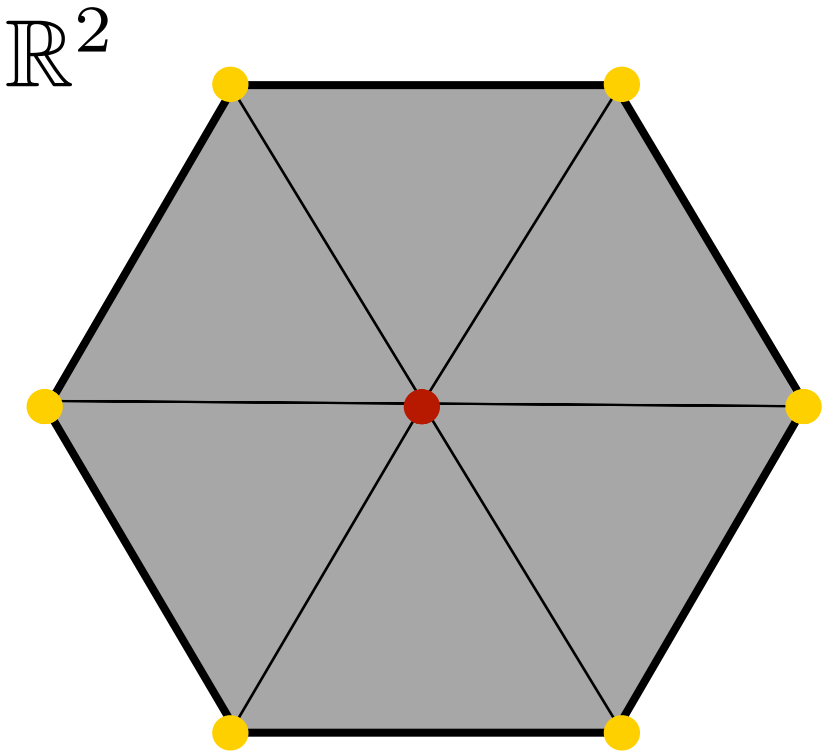
Remark: T_v is a parametrization of $st(v, S_T)$ in $\mathbb{R}^2$:



$st(v, S_T)$



T_v

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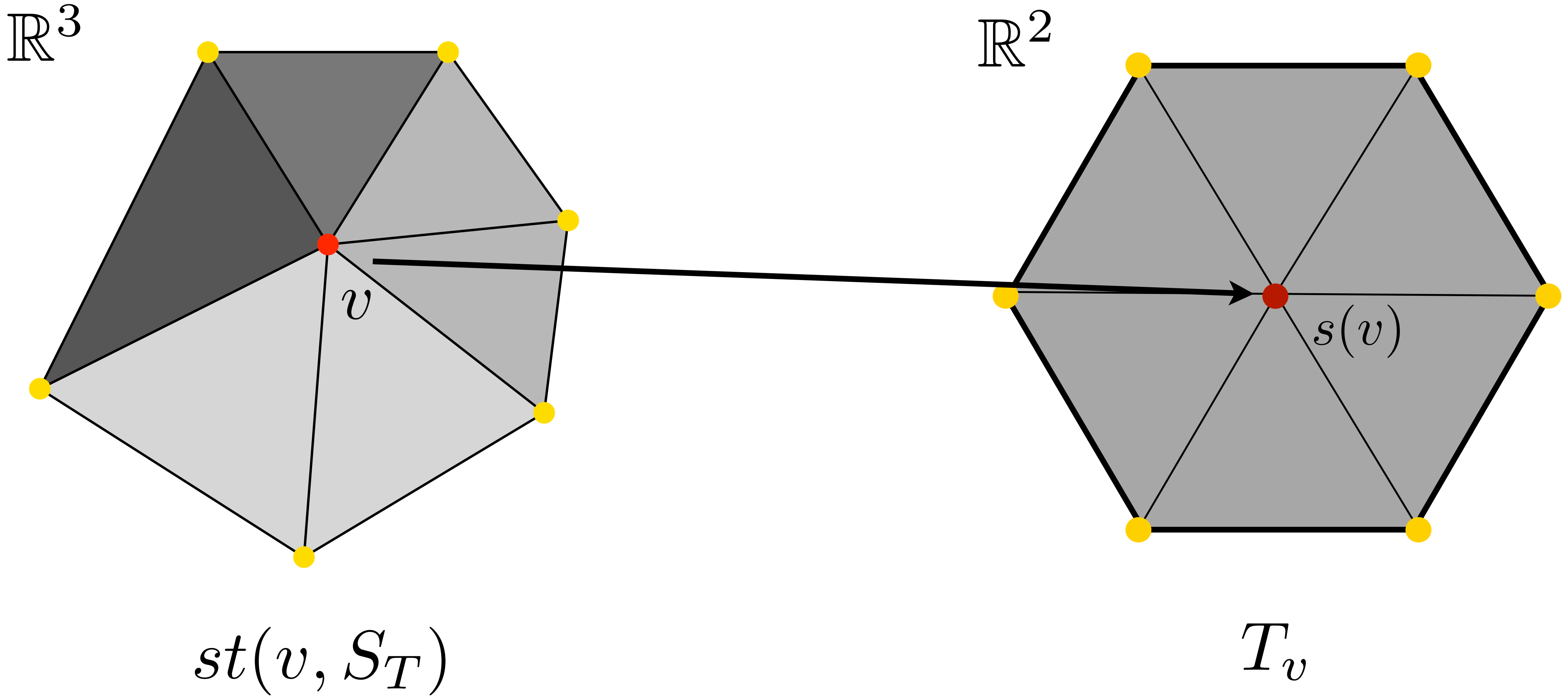

$$s : st(v, S_T) \to T_v$$

Building a Set of Gluing Data

Building a Set of Gluing Data

For each triangle σ of S_T and vertex v of σ, we define the **overlapping point**, $r_{v,\sigma}$, associated with $s(\sigma)$ in T_v, as follows:

Building a Set of Gluing Data

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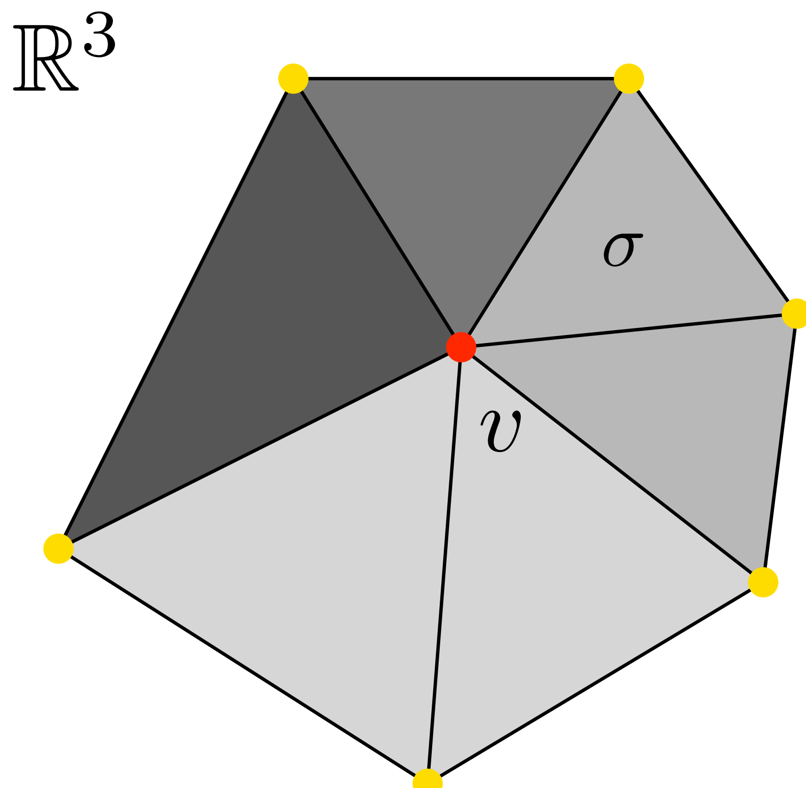

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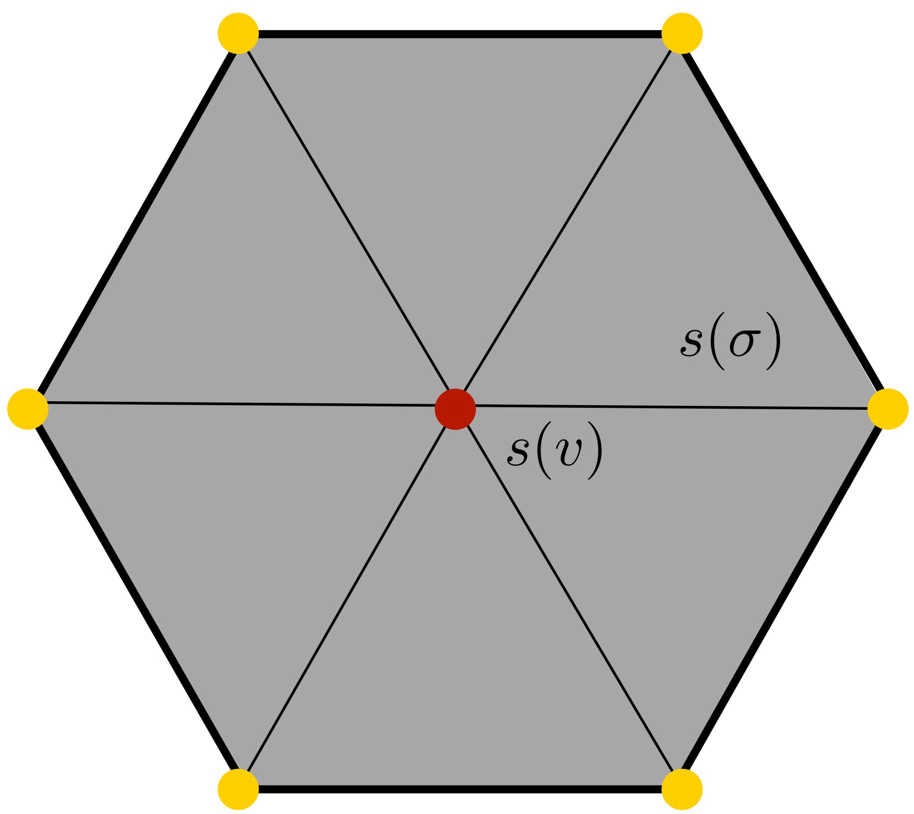

T_v

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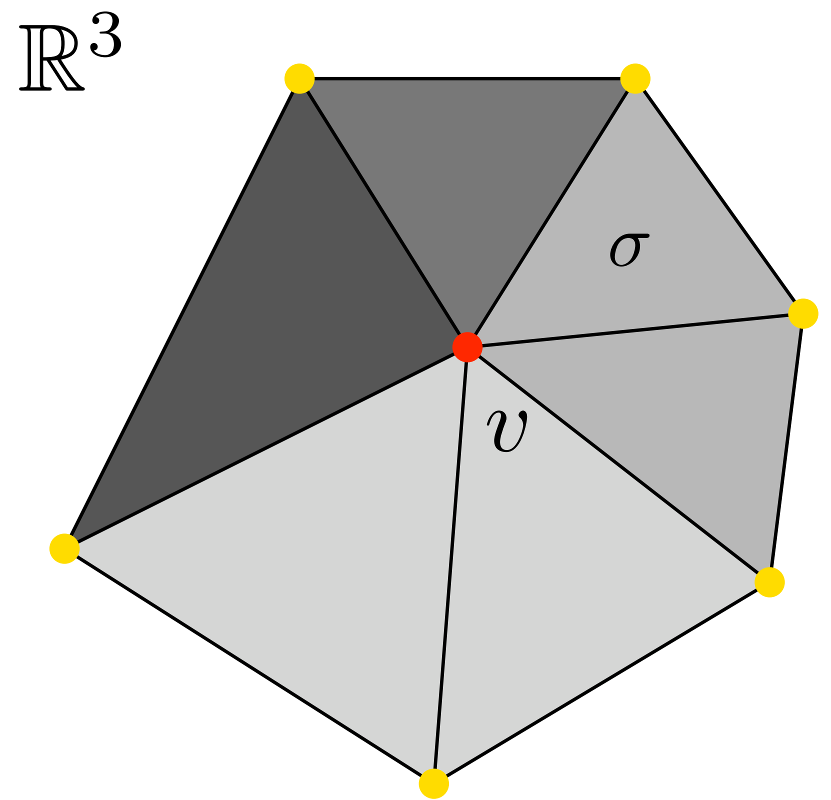
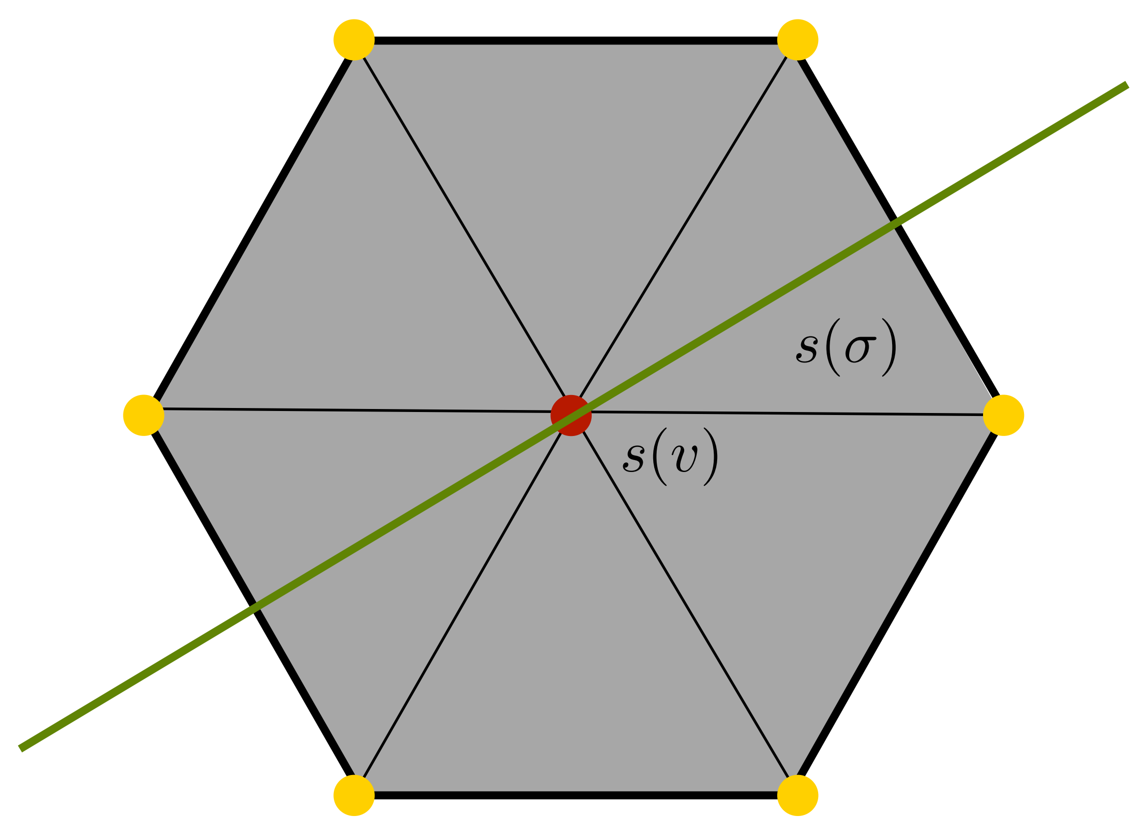


$st(v, S_T)$



T_v

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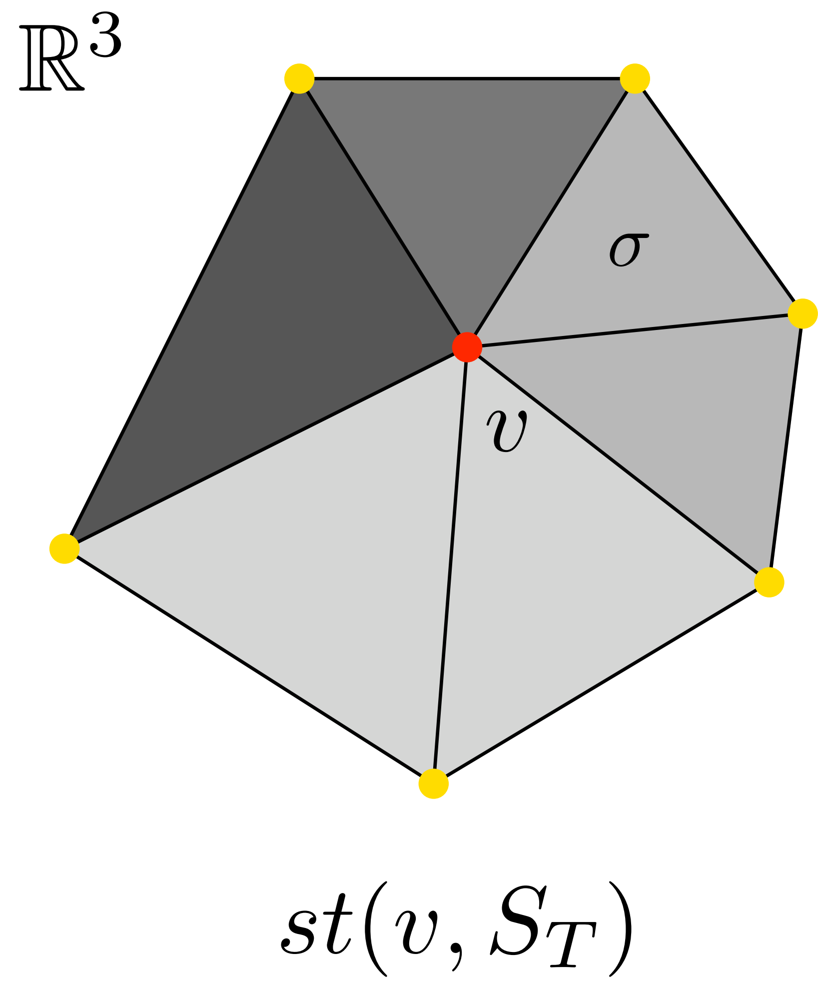

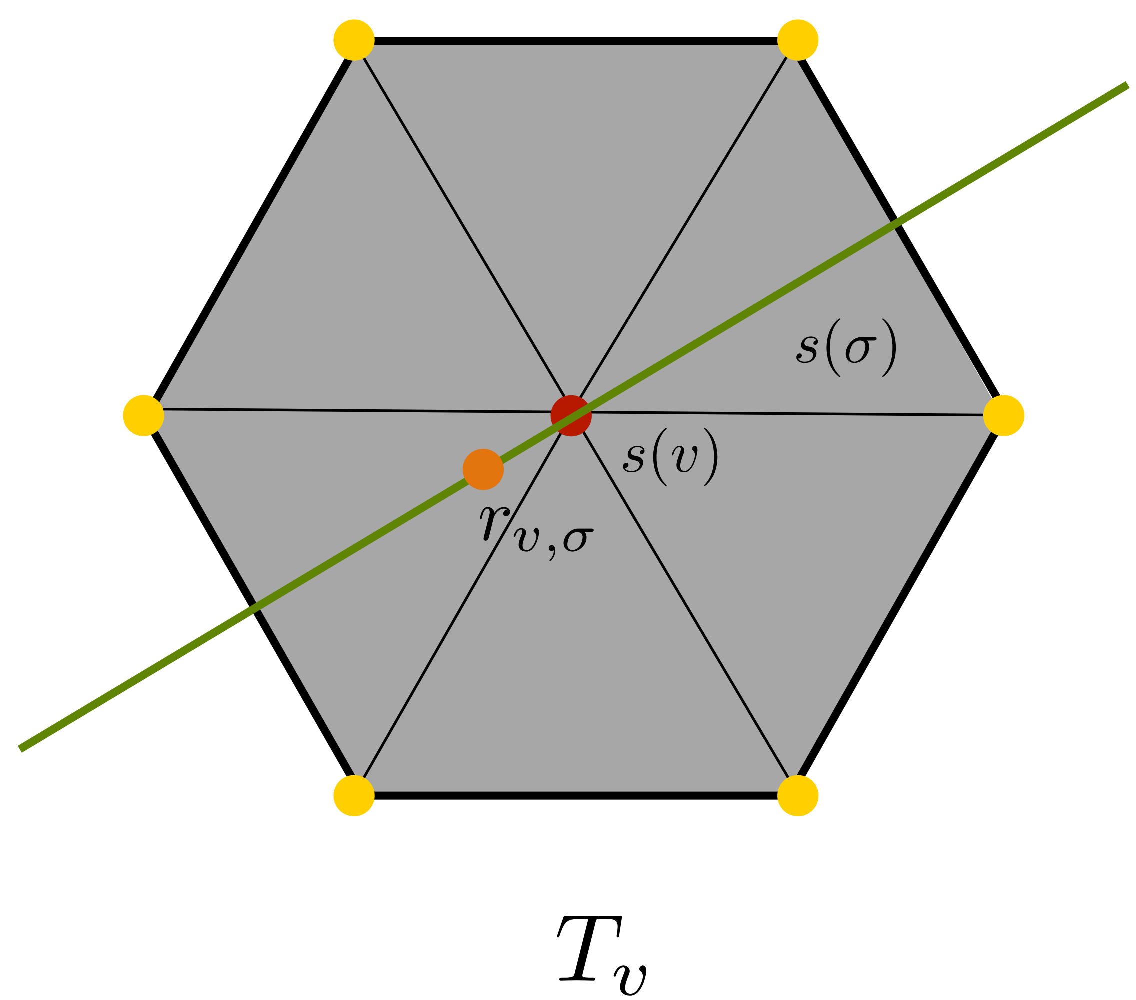

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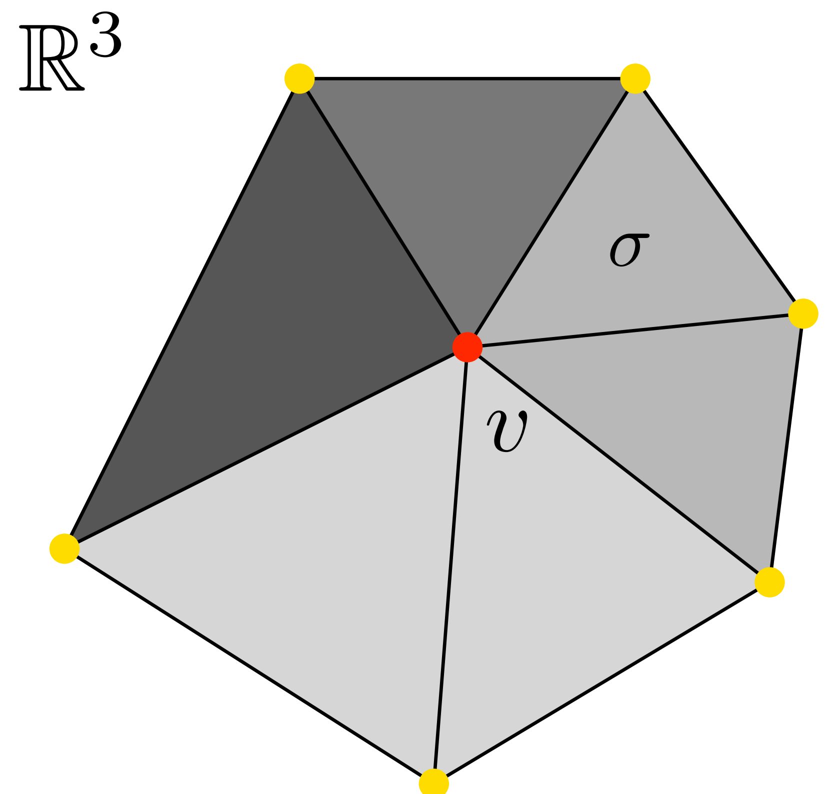

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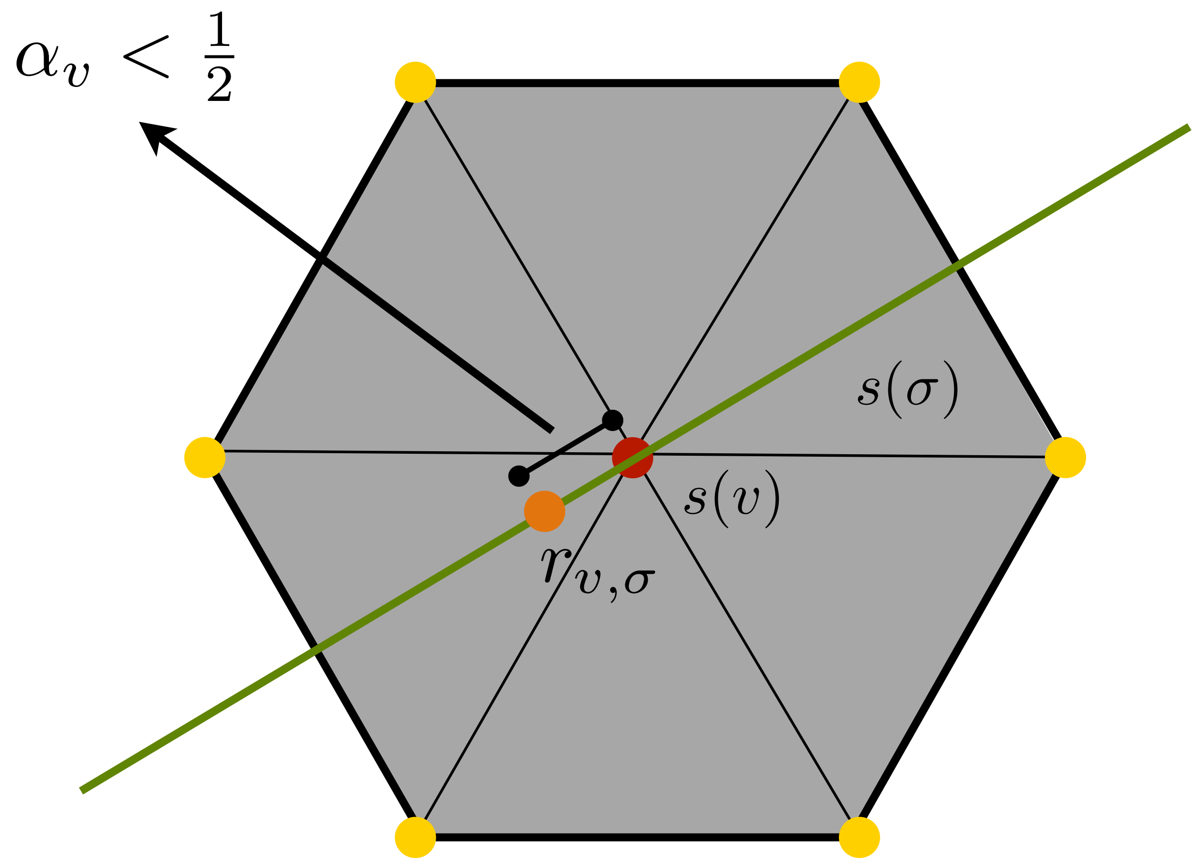

T_v

Building a Set of Gluing Data

Building a Set of Gluing Data

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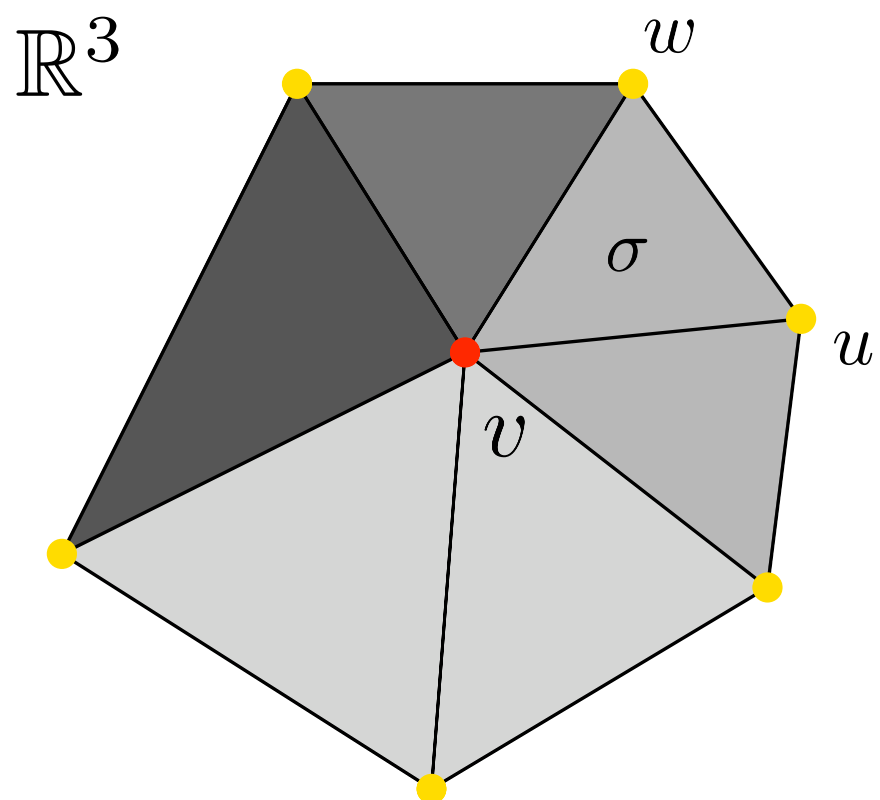

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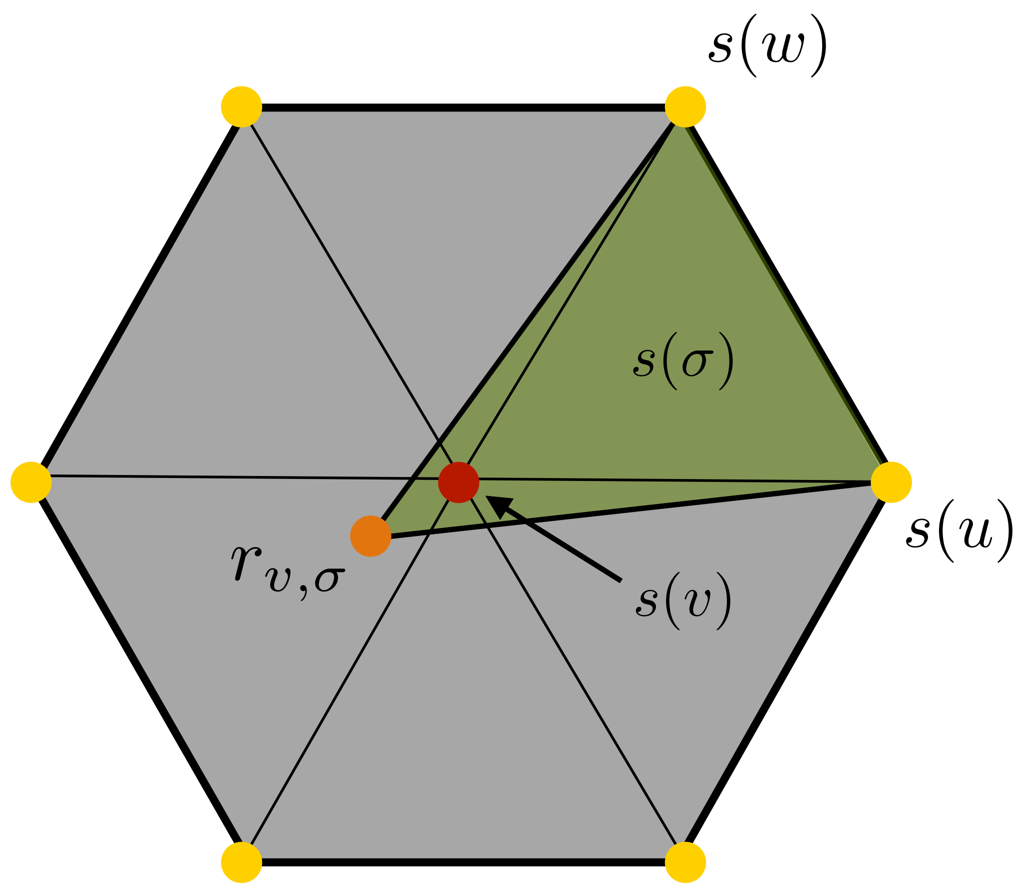

T_v

Building a Set of Gluing Data

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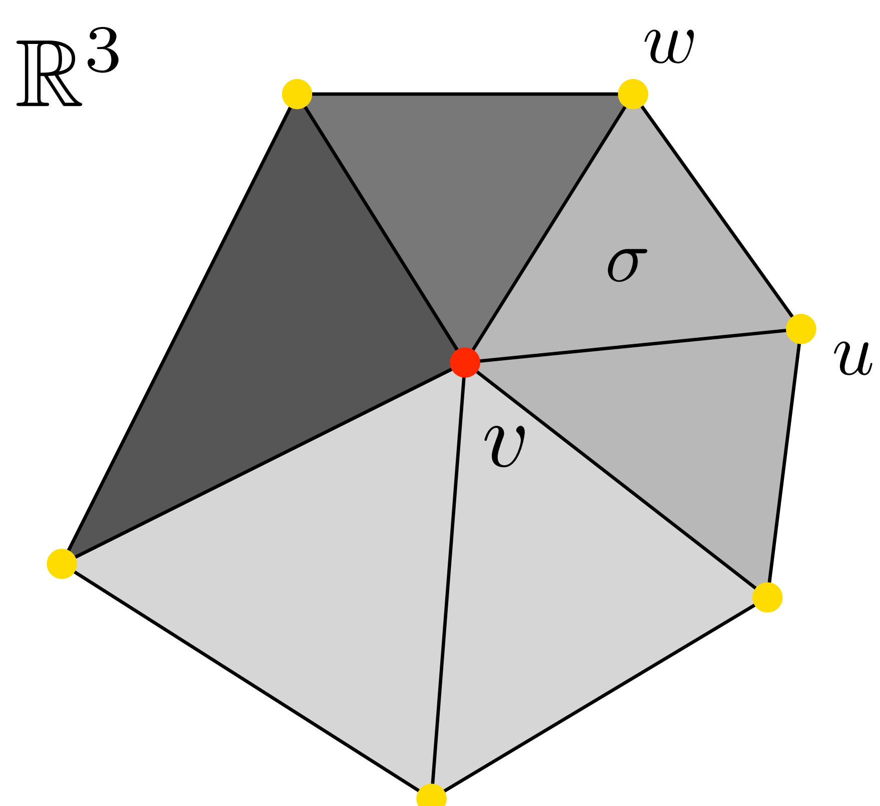

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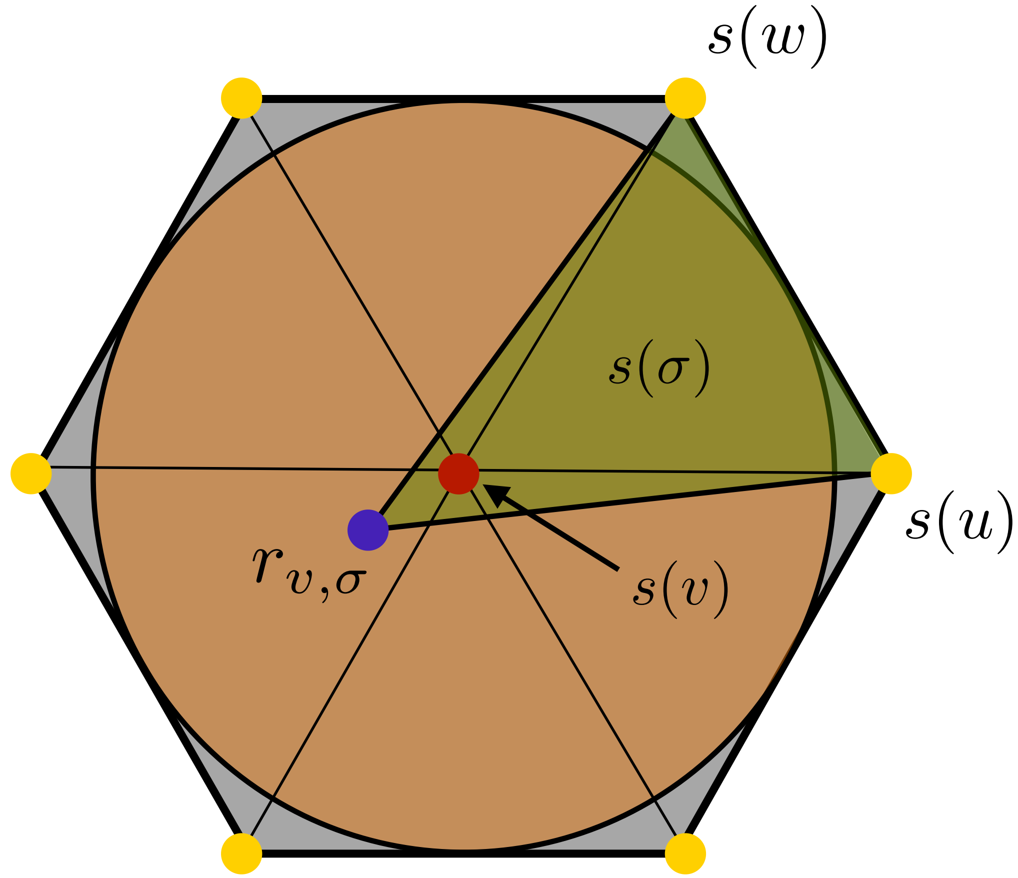

T_v

Building a Set of Gluing Data

Building a Set of Gluing Data

We let $\Omega_{(\sigma,v)}$ be

$$C_v \cap \mathrm{int}([r_{v,\sigma}, s(u), s(w)])\,,$$

where $\mathrm{int}([r_{v,\sigma}, s(u), s(w)])$ is the interior of $[r_{v,\sigma}, s(u), s(w)]$.

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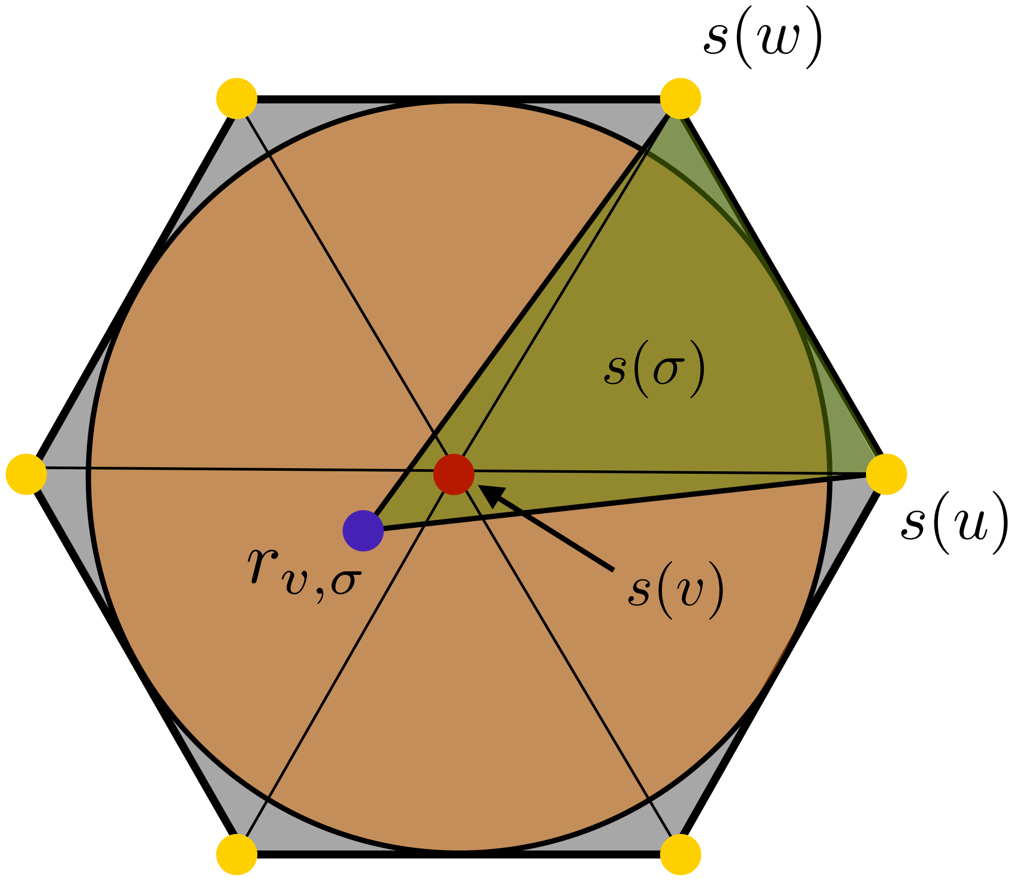

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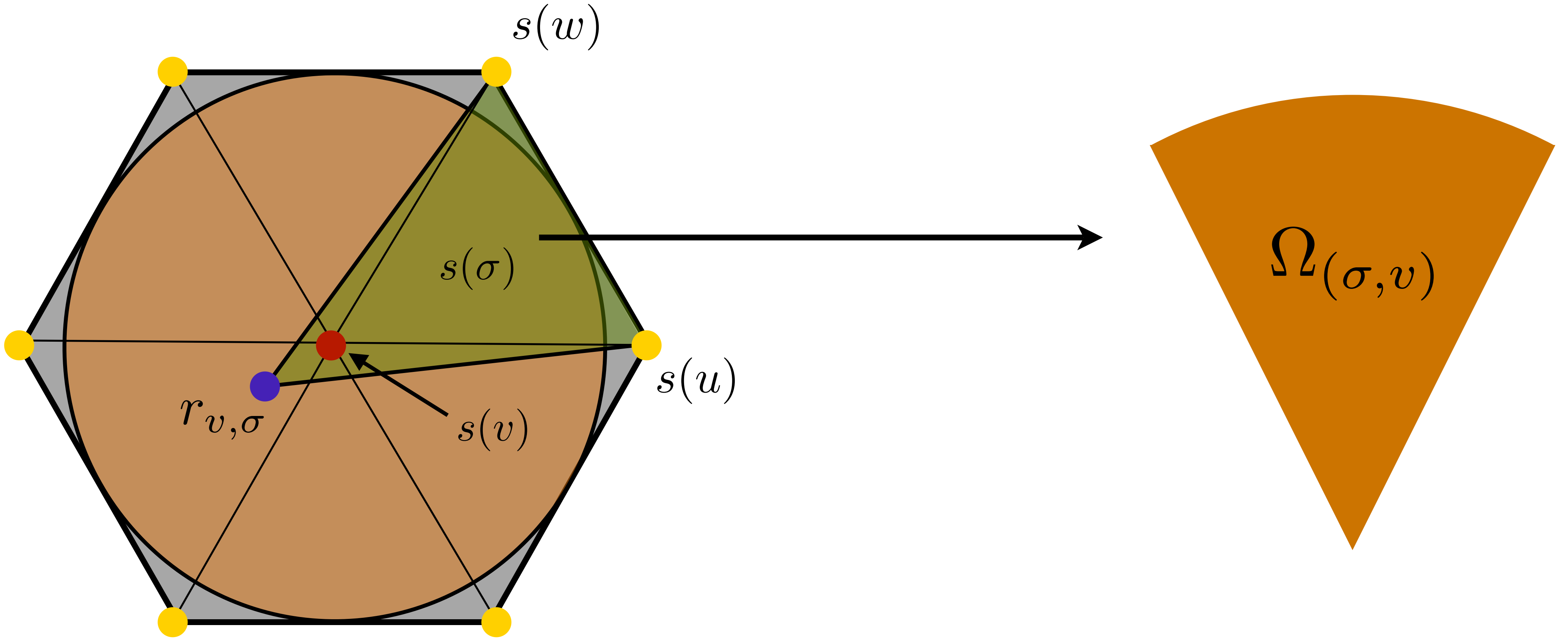


Building a Set of Gluing Data

Remark:

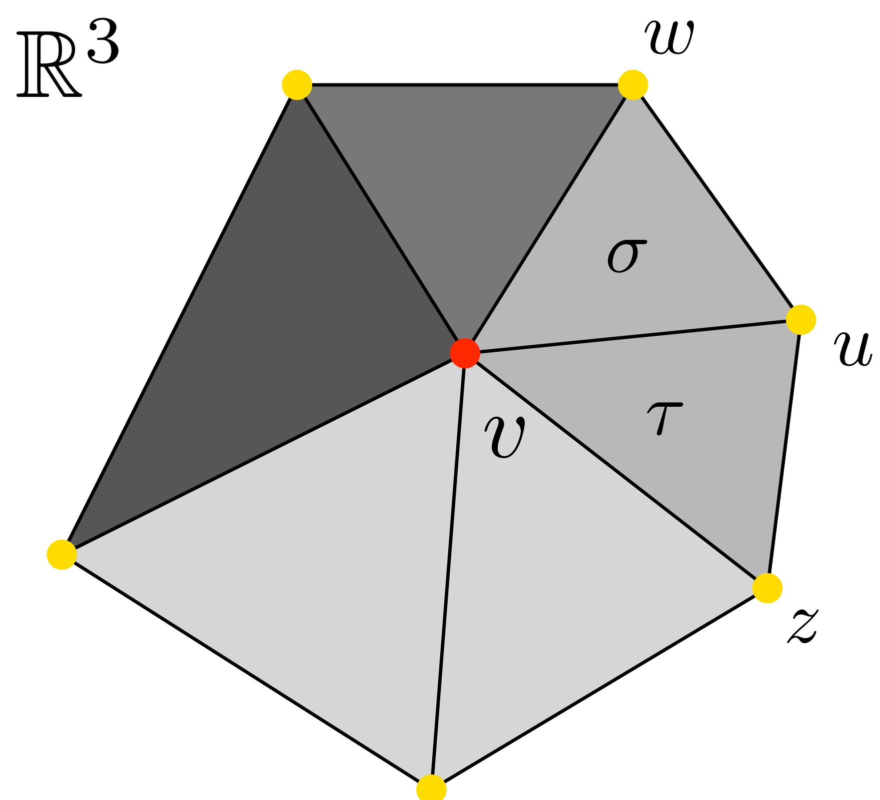
From Jean Gallier's lecture, we should have

$$\Omega_{(\sigma,v)} \cap \Omega_{(\tau,u)} = \emptyset \, ,$$

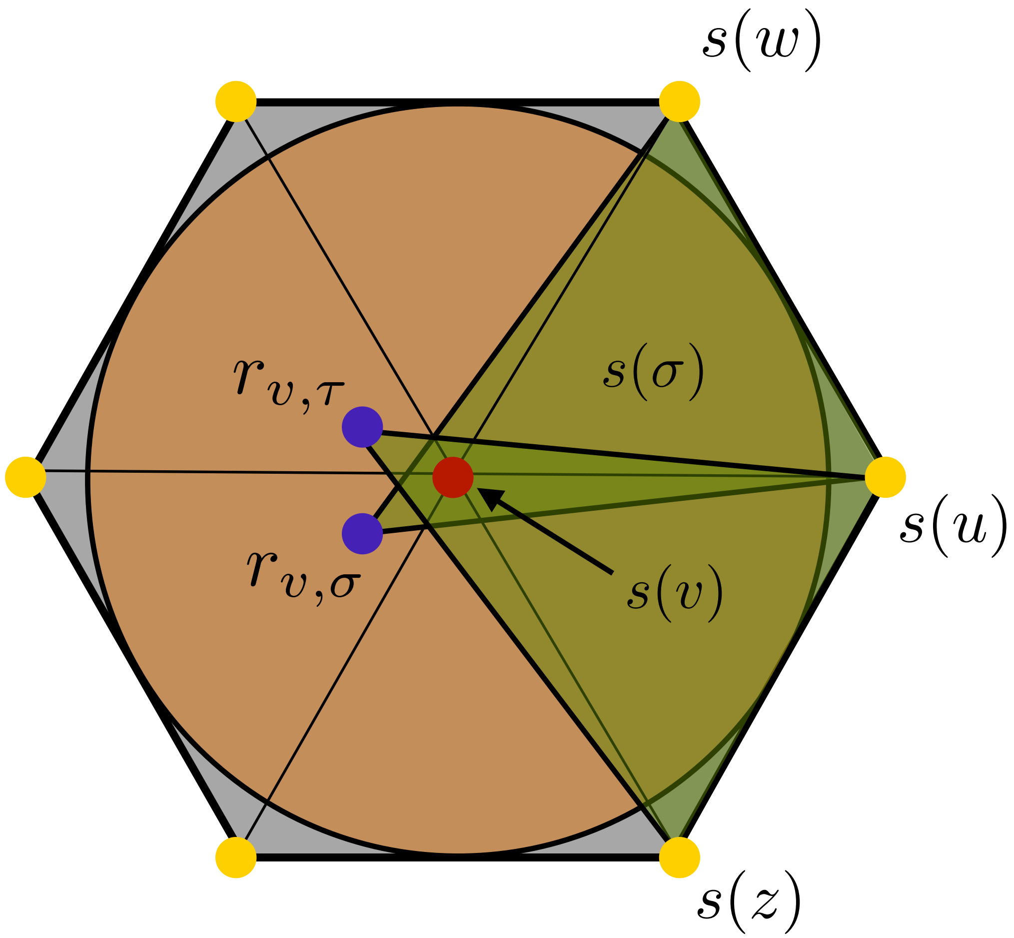
for any two pairs, (σ, v) and (τ, u), in I. Did I make it right?

Building a Set of Gluing Data

Building a Set of Gluing Data

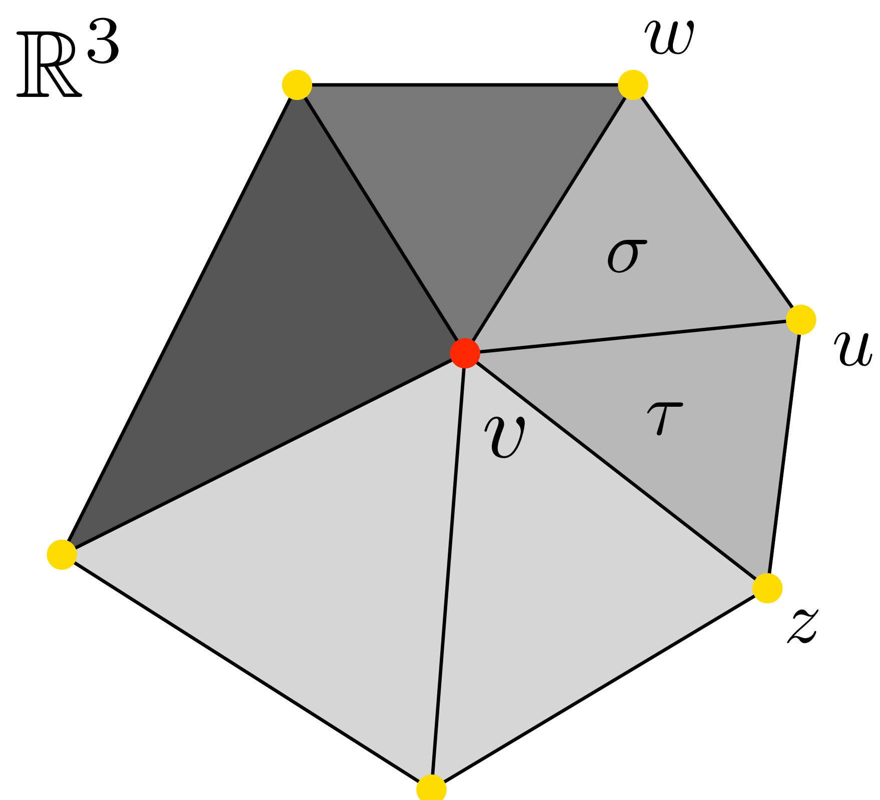


$st(v, S_T)$



T_v

Building a Set of Gluing Data



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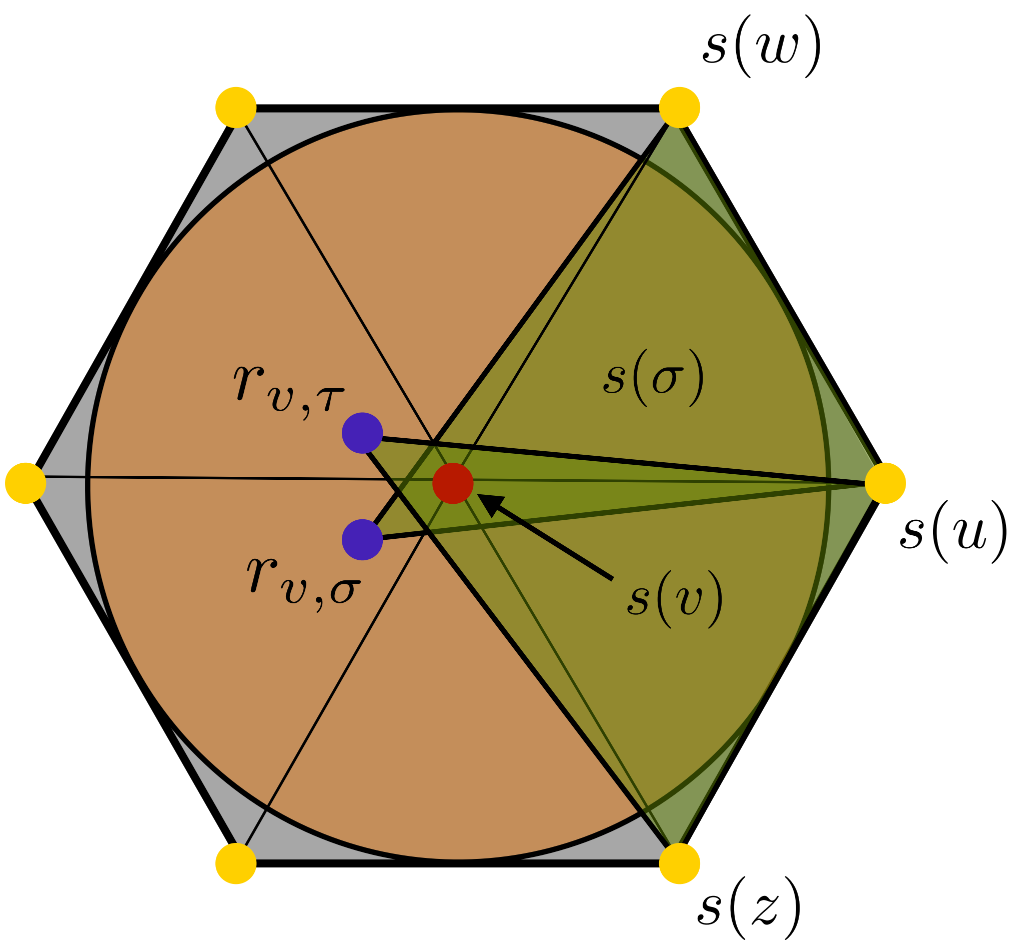


T_v

Clearly, $\Omega_{(\sigma,v)} \cap \Omega_{(\tau,v)} \neq \emptyset$.

Building a Set of Gluing Data

Building a Set of Gluing Data

So, I did NOT make it right.

Building a Set of Gluing Data

So, I did NOT make it right.

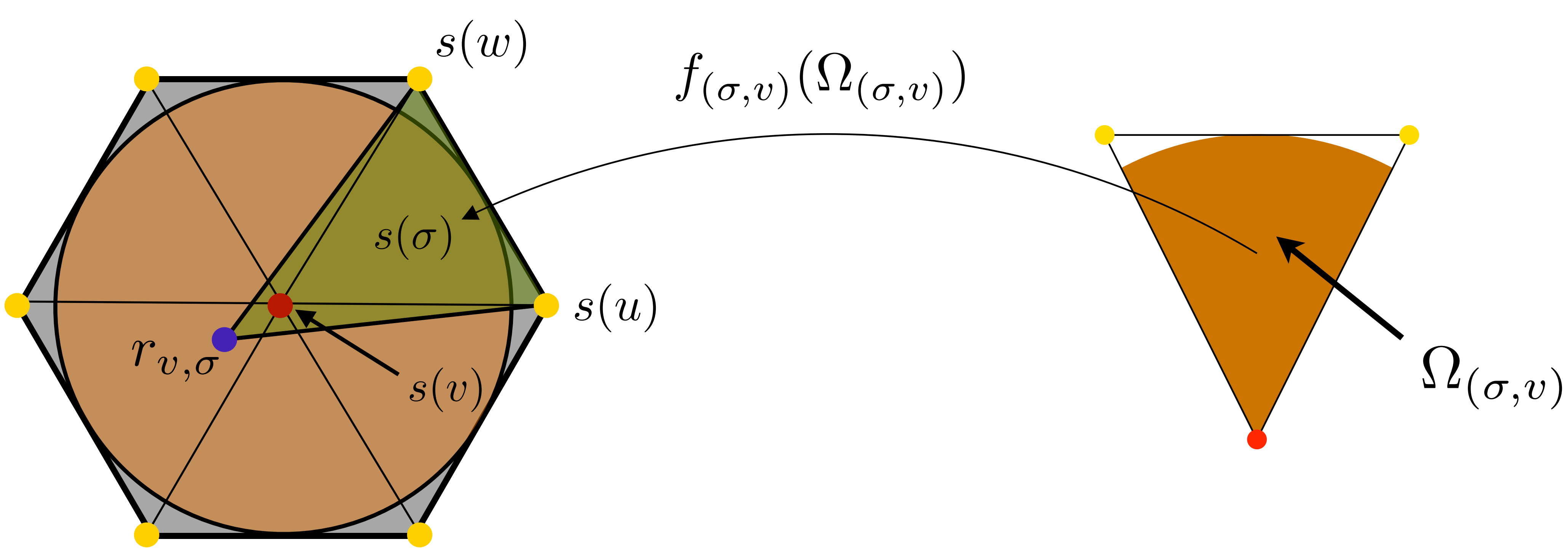
What now?

Building a Set of Gluing Data

So, I did NOT make it right.

What now?

We can *fix* that by letting $\Omega_{(\sigma,v)}$ be a set inside a triangle which is the inverse image of $[r_{v,\sigma}, s(u), s(w)]$ under a rigid transformation!



Building a Set of Gluing Data

Building a Set of Gluing Data

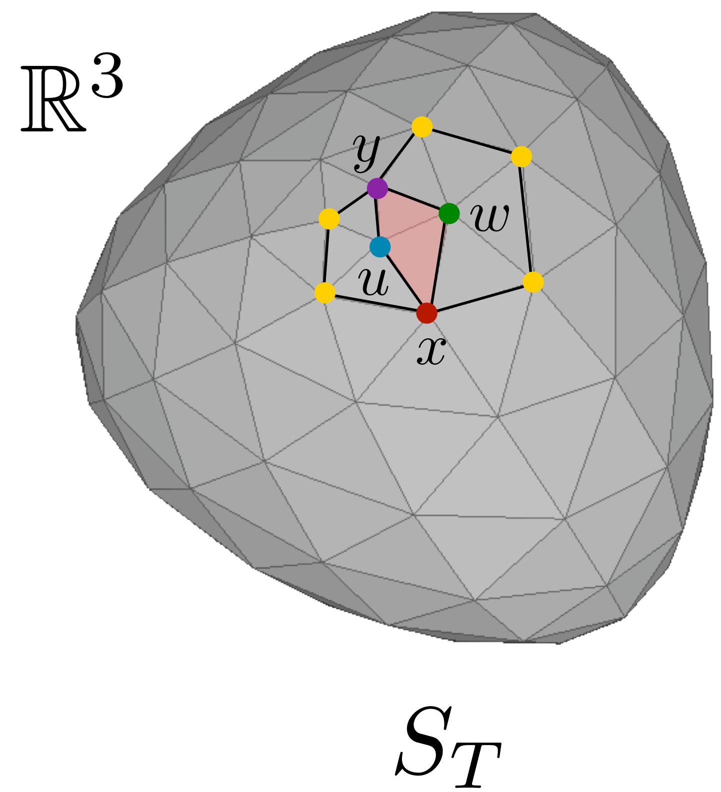
Since I is a finite set and the "enclosing" triangles are compact, we can certainly separate each p-domain from the others in $\mathbb{R}^2$.

Building a Set of Gluing Data

Gluing domains

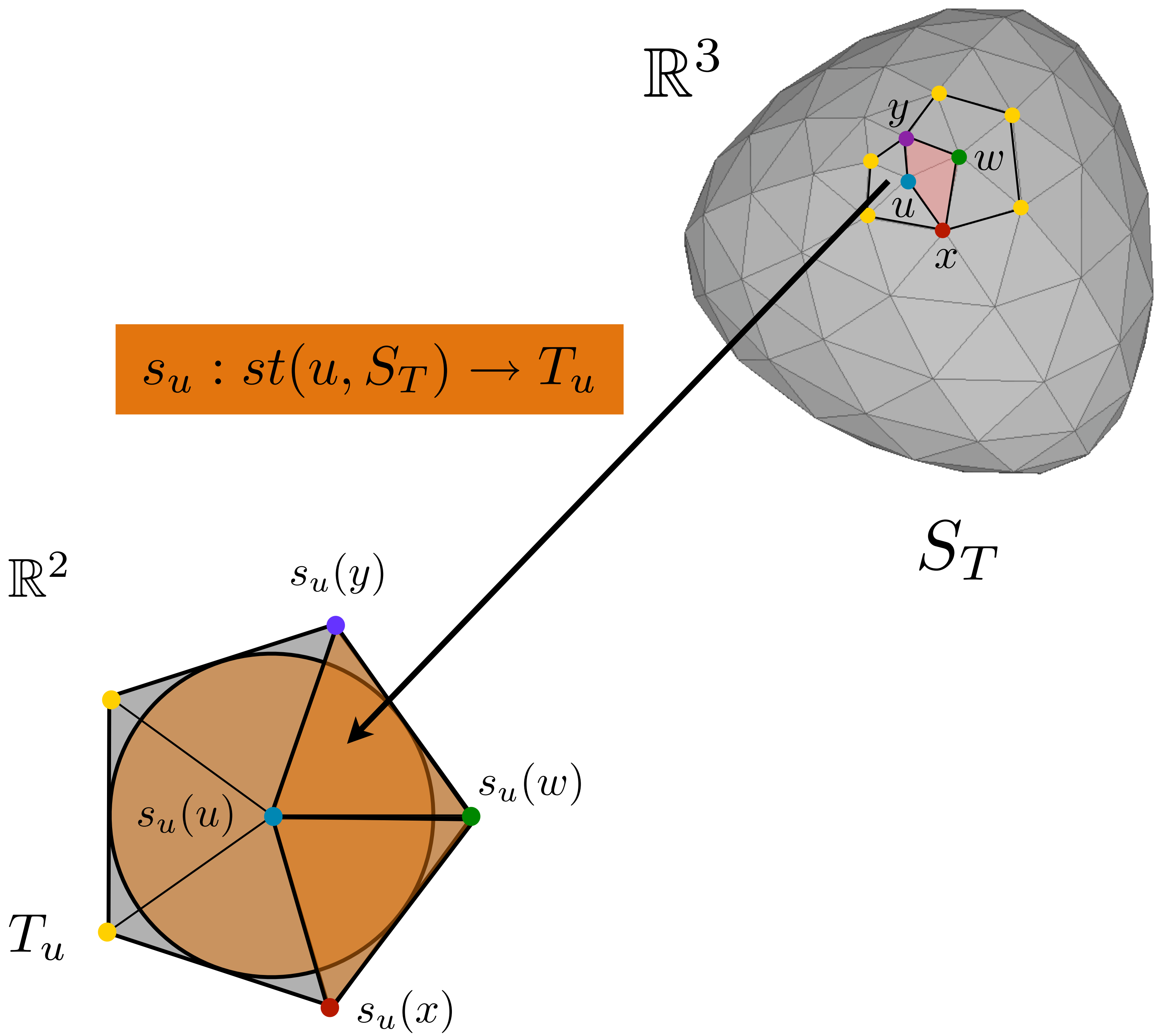
Building a Set of Gluing Data

Gluing domains



Building a Set of Gluing Data

Gluing domains



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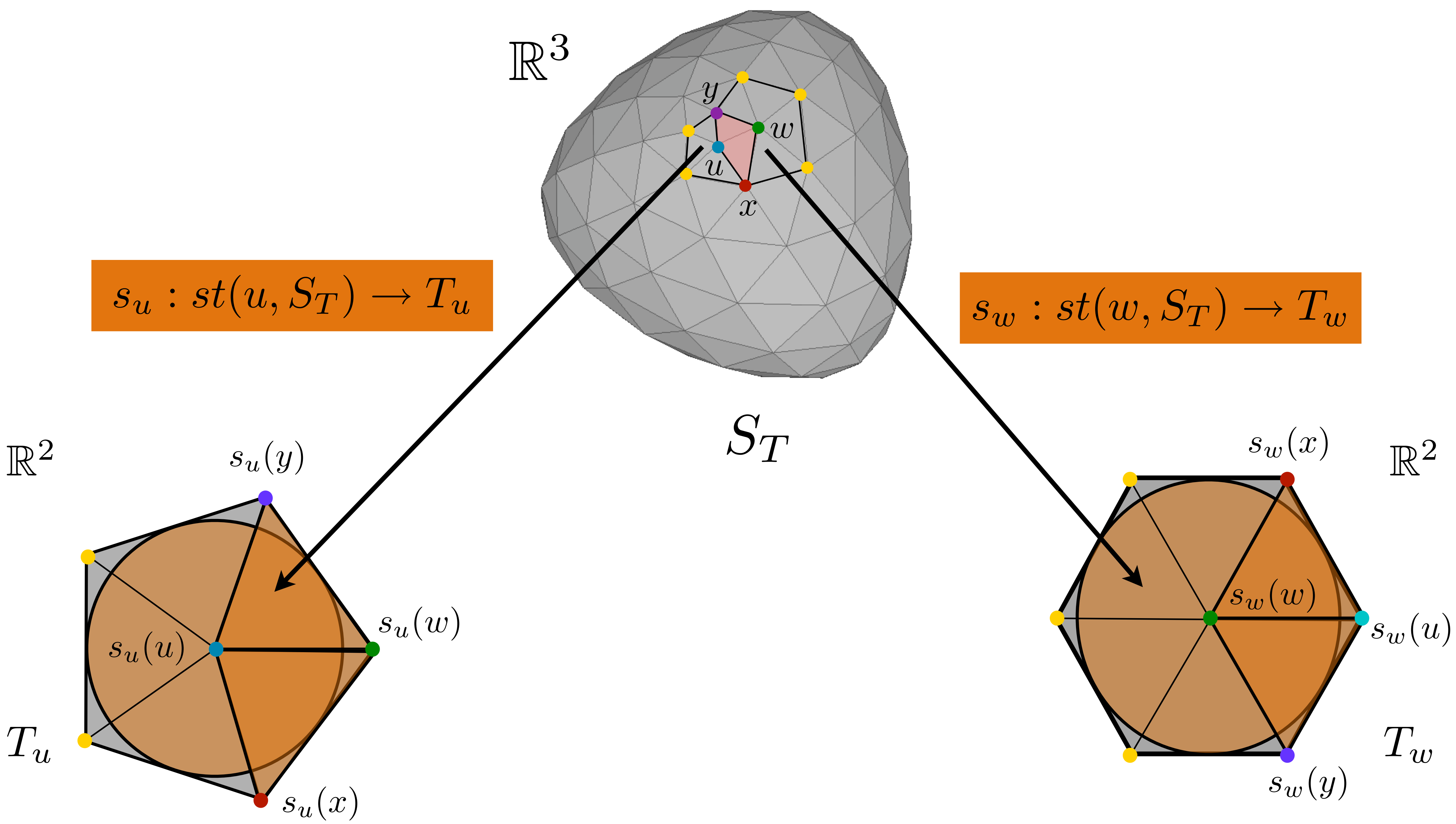

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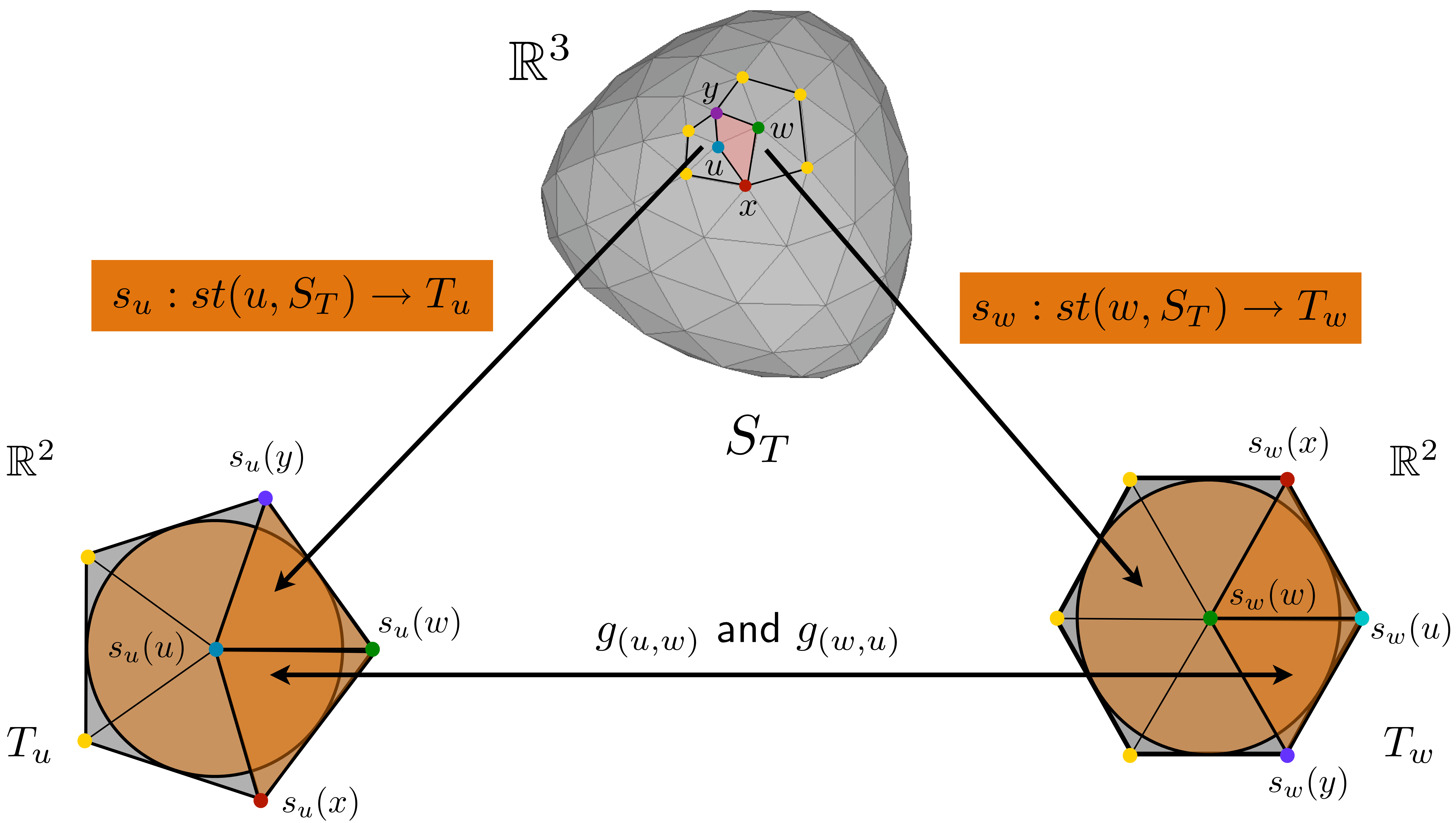
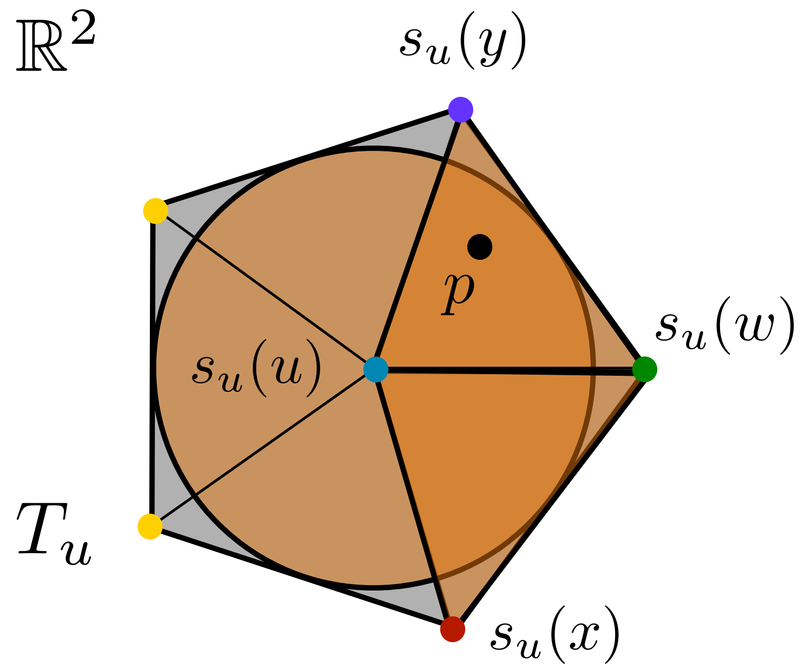


Building a Set of Gluing Data

Building a Set of Gluing Data

Let p be a point in the region $C_u \cap [s_u(u), s_u(x), s_u(w), s_u(y)]$.



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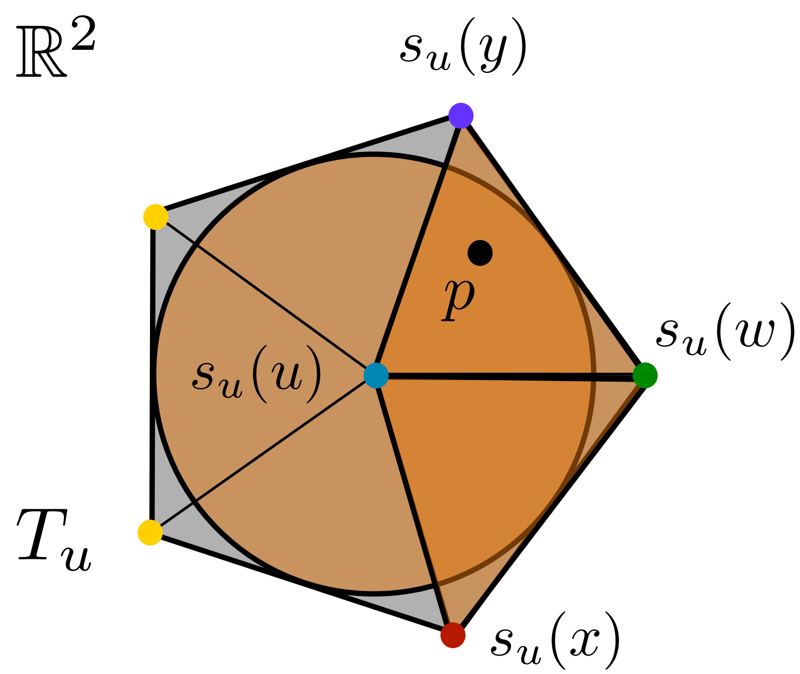


Let (θ, r) be the polar coordinates of point p with respect to the local coordinate system of P_u (i.e., origin at $s_u(u) = (0,0)$).

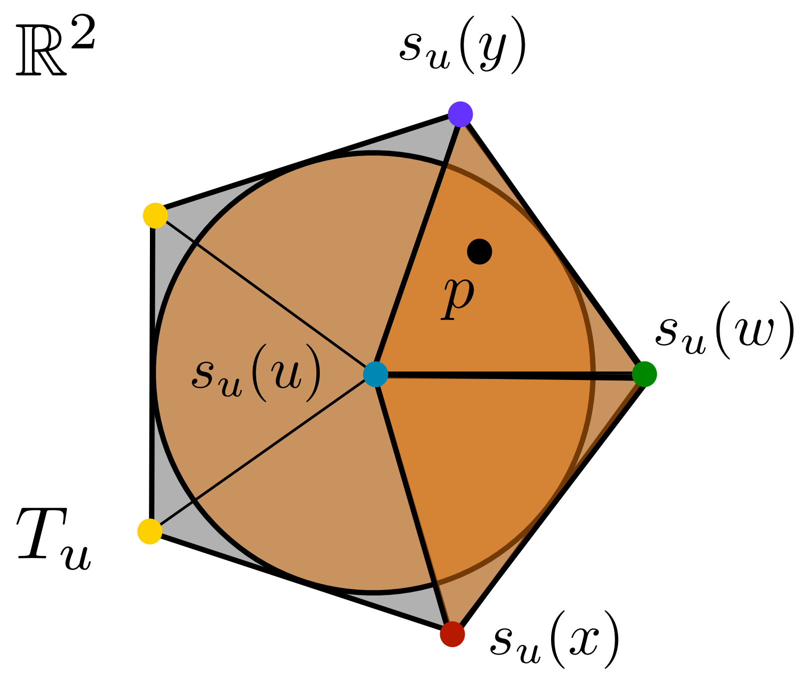
Building a Set of Gluing Data

Building a Set of Gluing Data

Let $g_u : [0, 2\pi) \times \mathbb{R}_+ \to [0, 2\pi) \times \mathbb{R}_+$ be the map

$$g_u(p) = g_u((\theta, r)) = \left(\frac{6}{m_u} \cdot \theta \, , \, \frac{\cos(\frac{\pi}{6})}{\cos(\frac{\pi}{m_u})} \cdot r \right) ,$$

where m_u is the degree of u.



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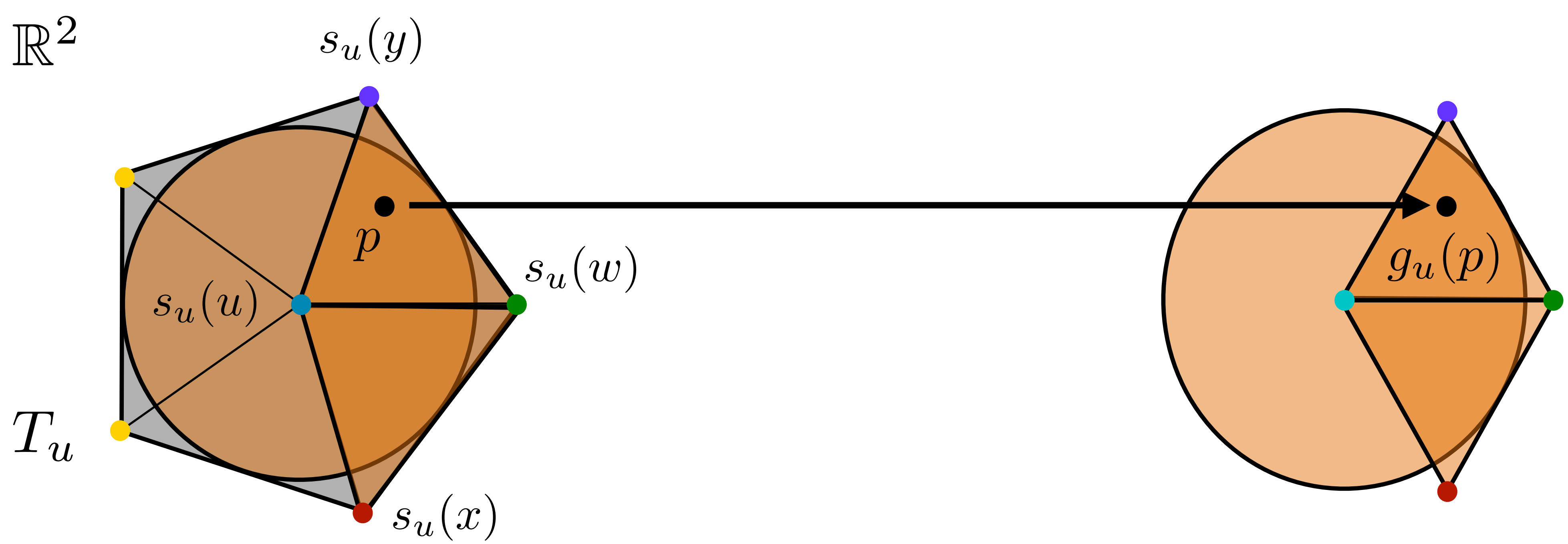

Building a Set of Gluing Data

Building a Set of Gluing Data

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $h(p) = h((x, y)) = (1 - x, -y)$:

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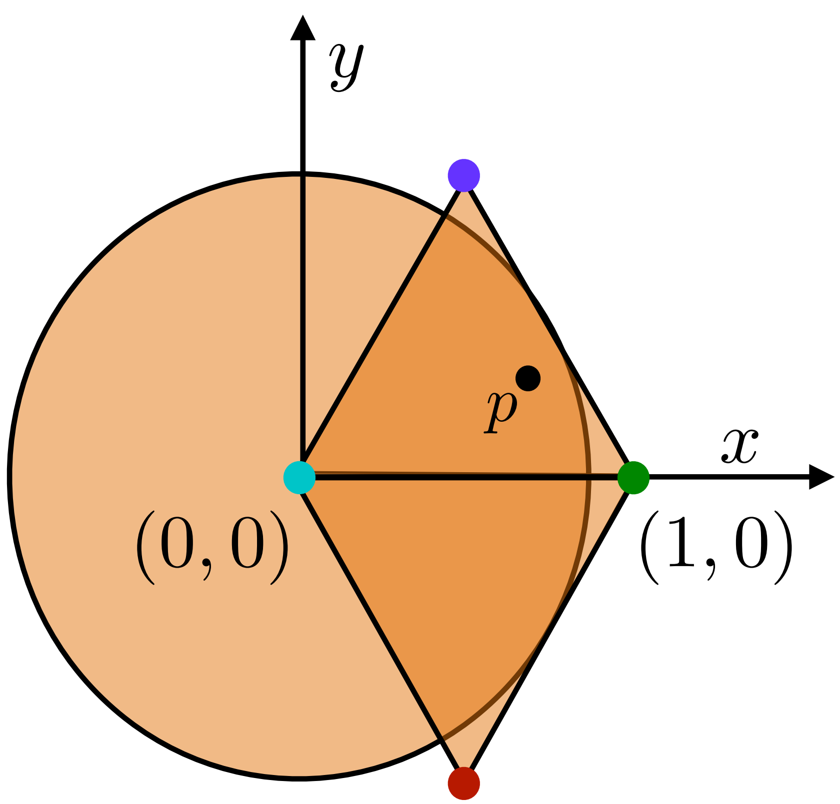

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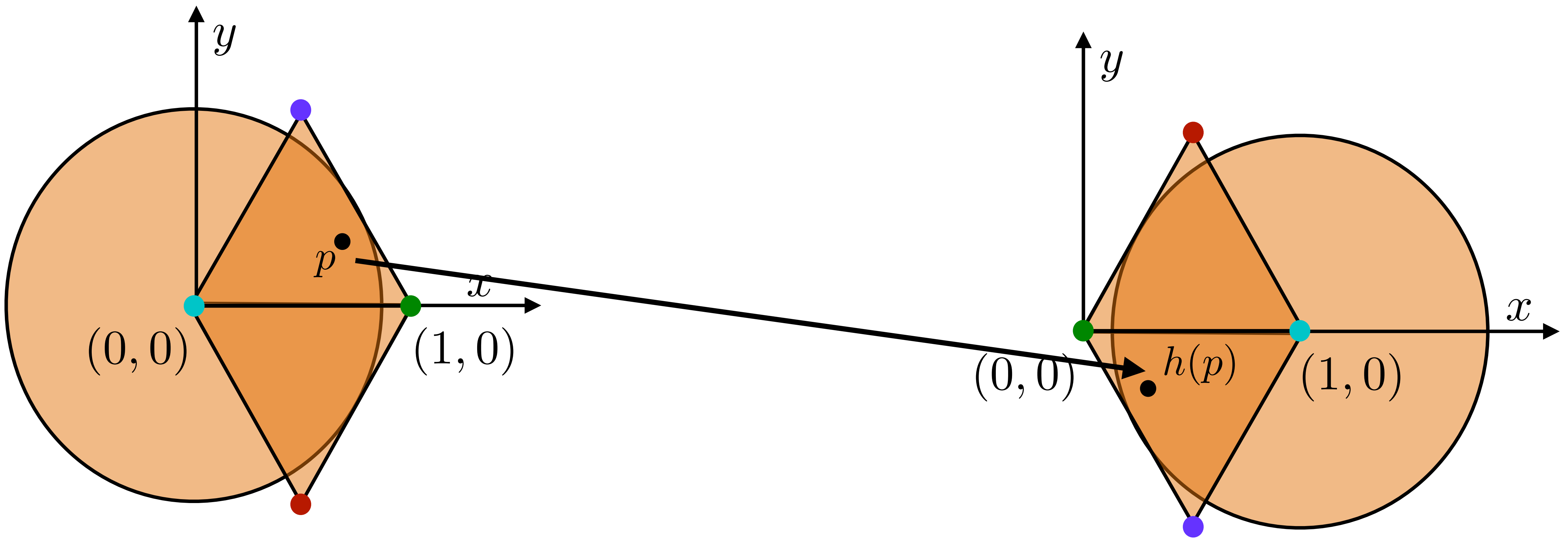

Building a Set of Gluing Data

Building a Set of Gluing Data

Finally, we define $g_{(u,w)} : [0, 2\pi) \times \mathbb{R}_+ \to [0, 2\pi) \times \mathbb{R}_+$ as

$$g_{(u,w)}(p) = g_{(u,w)}((\theta, r)) = g_w^{-1} \circ h \circ g_u((\theta, r)) \, .$$

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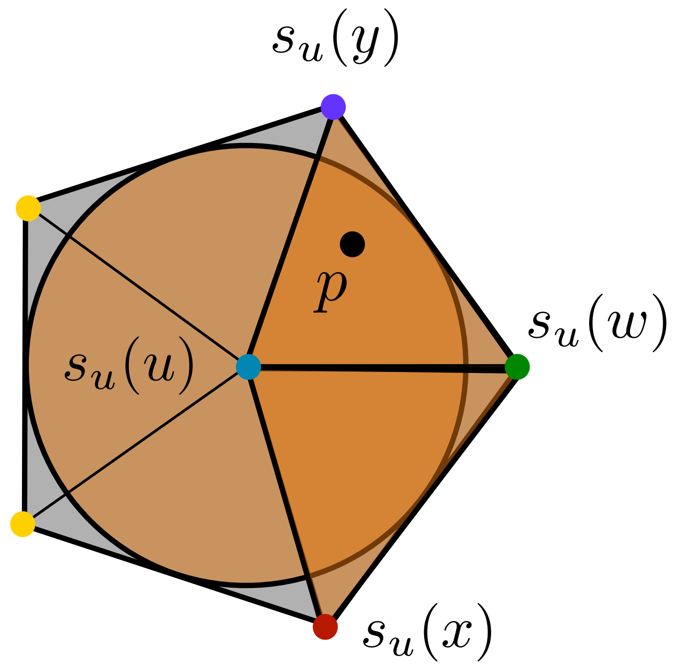

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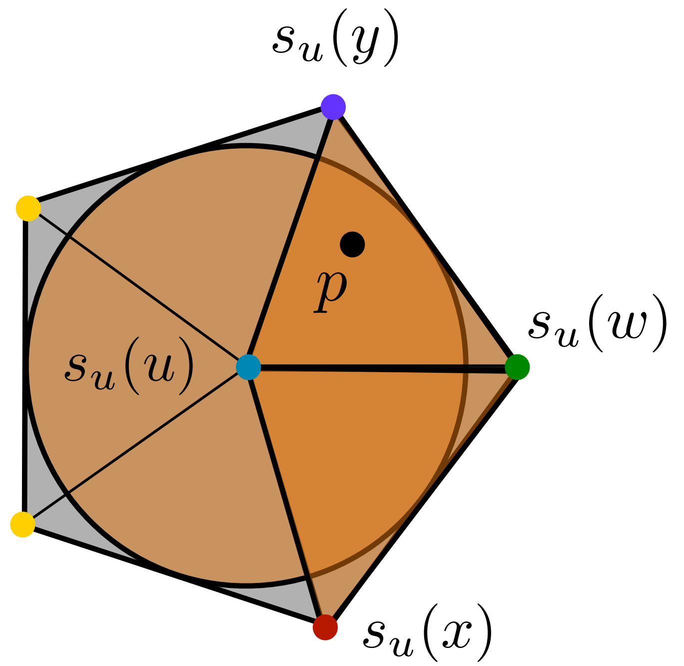

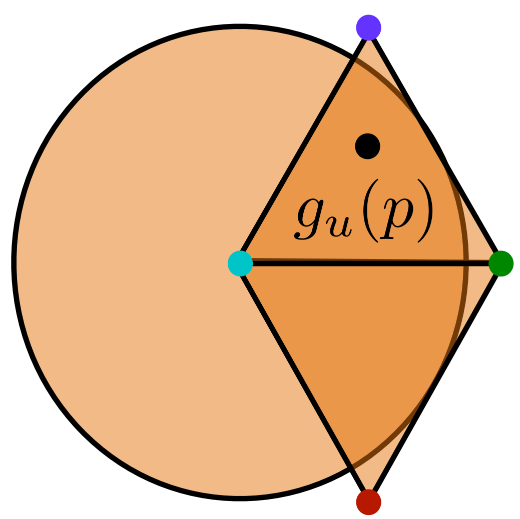

Building a Set of Gluing Data

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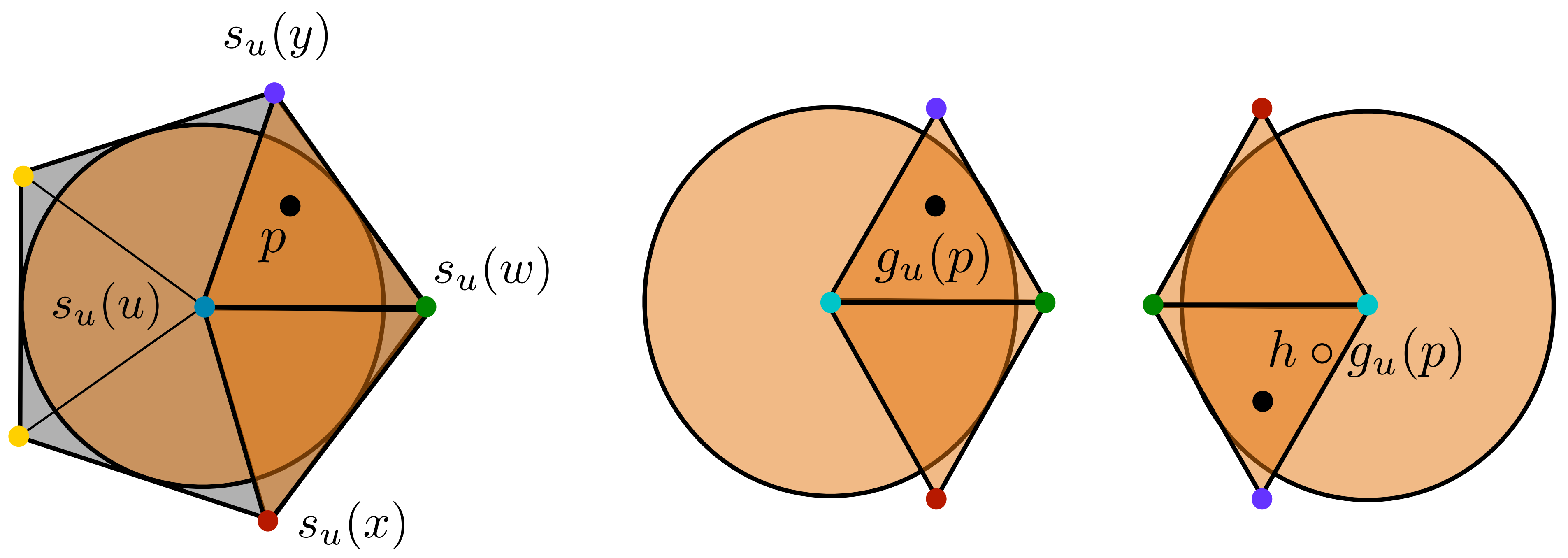

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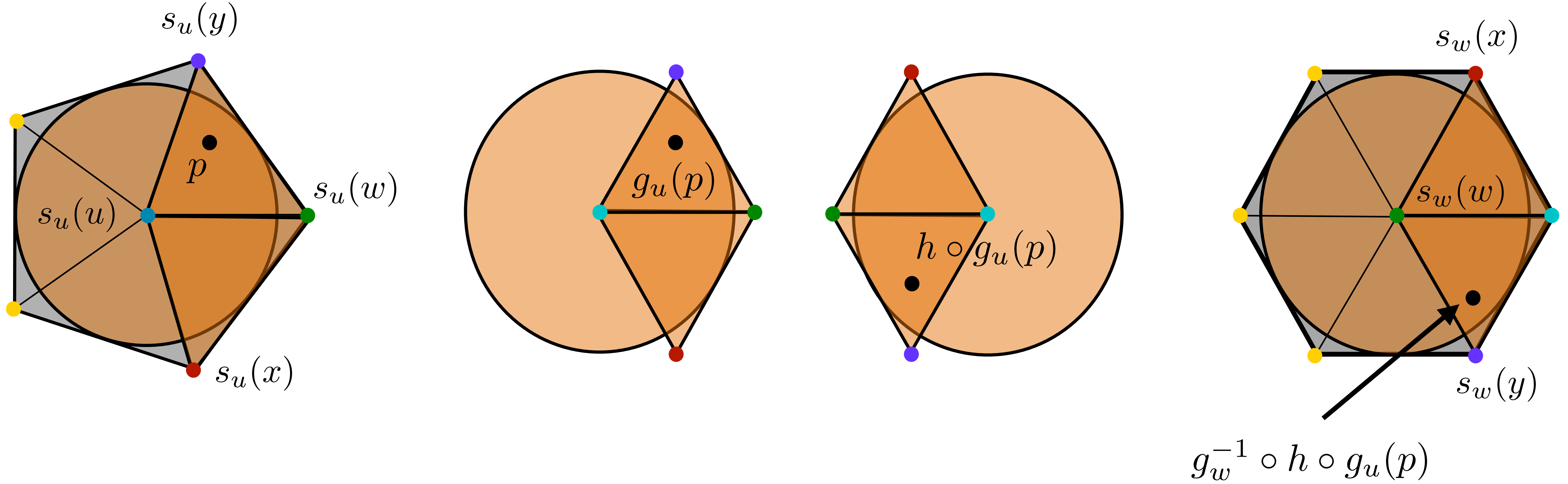

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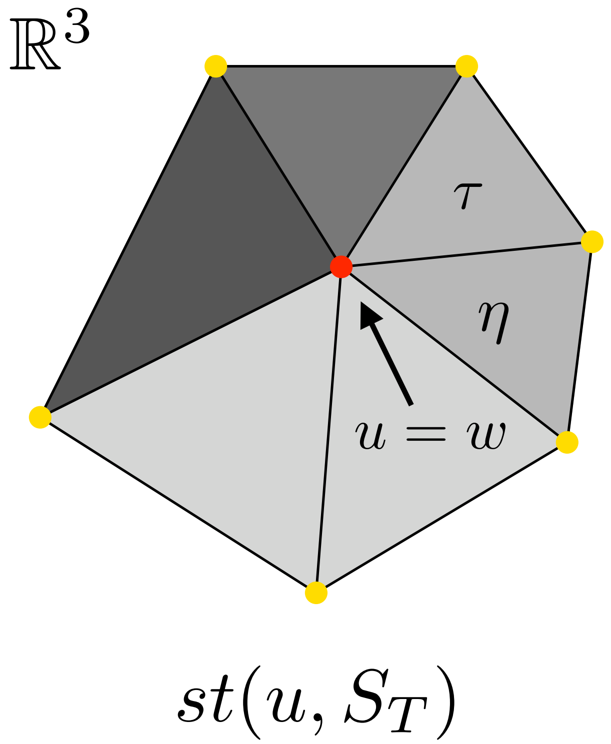

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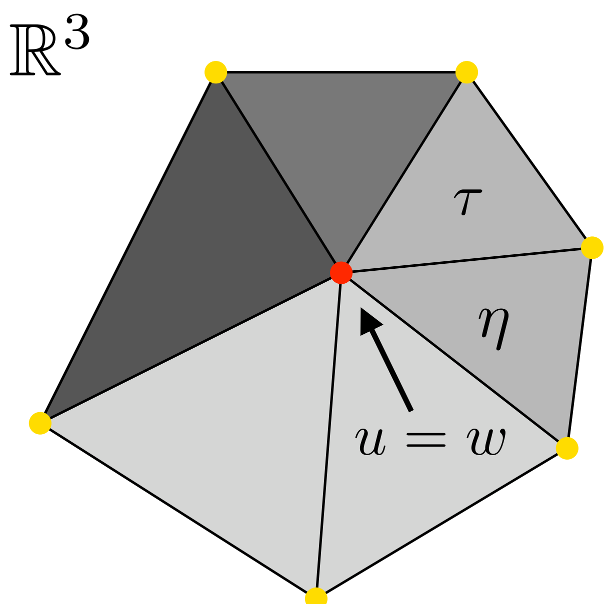

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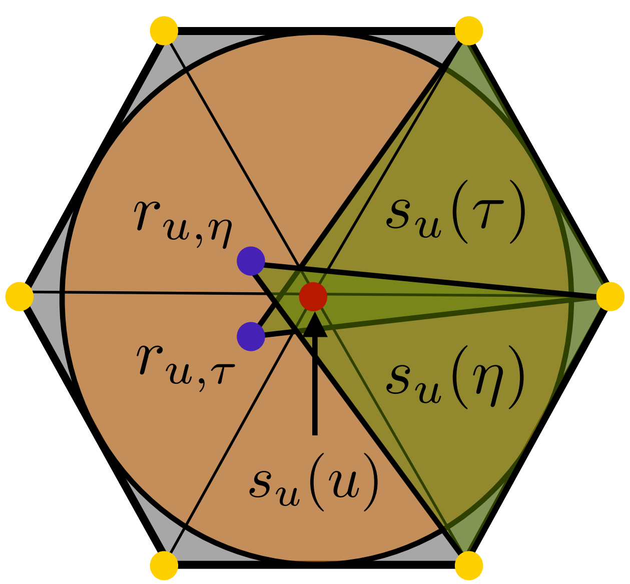

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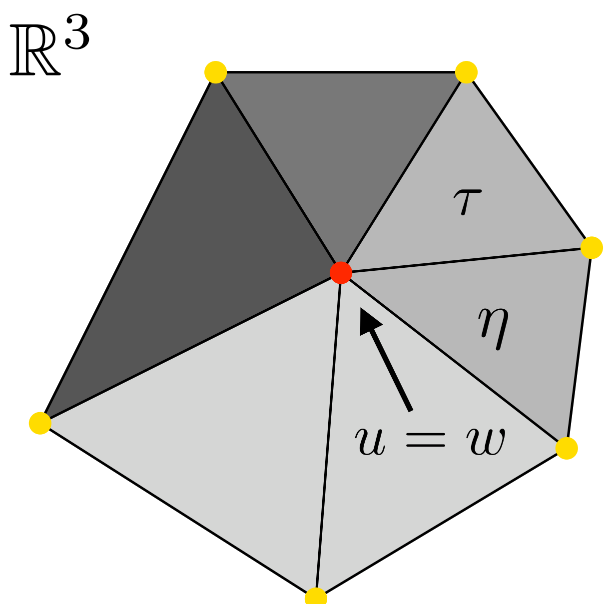

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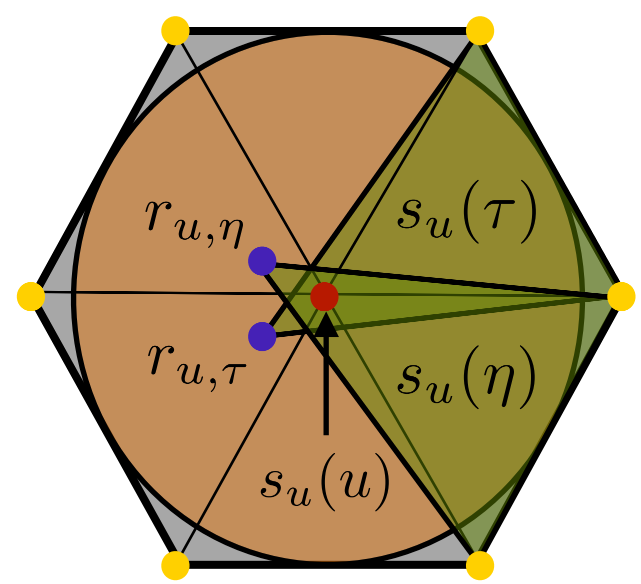

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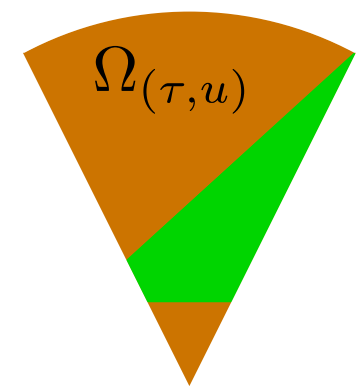

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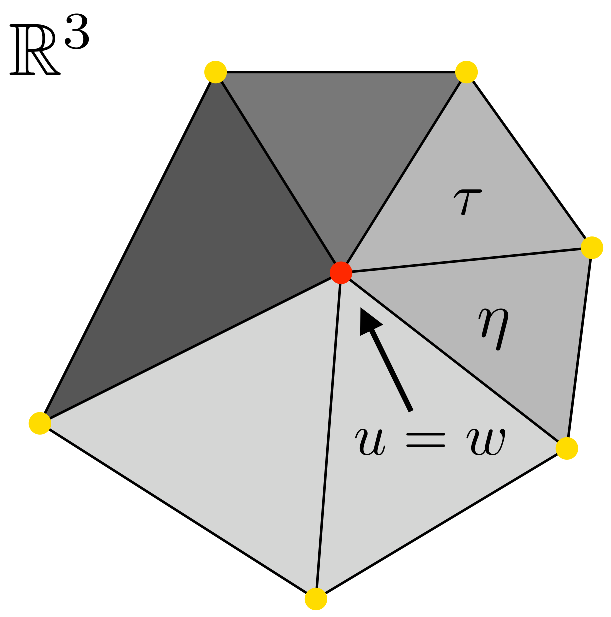

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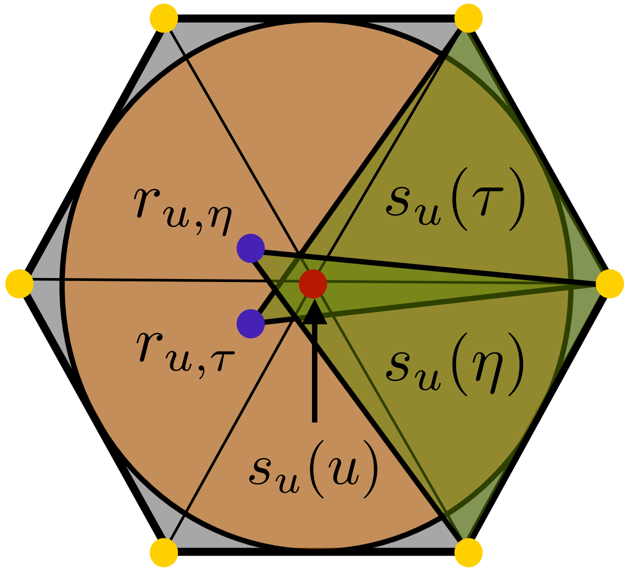

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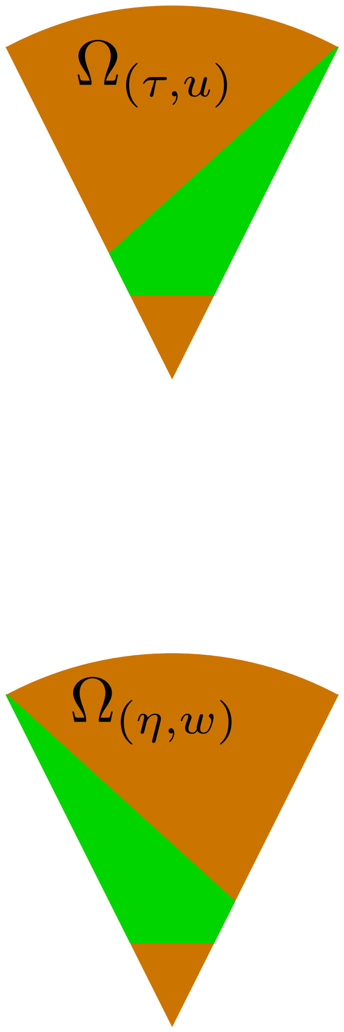

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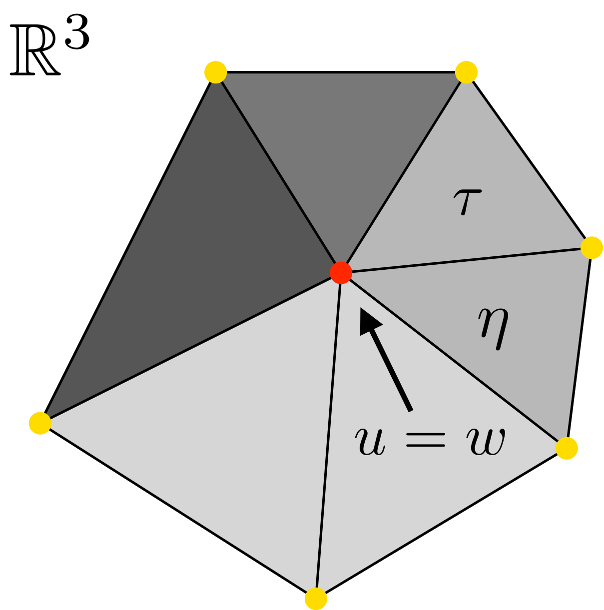

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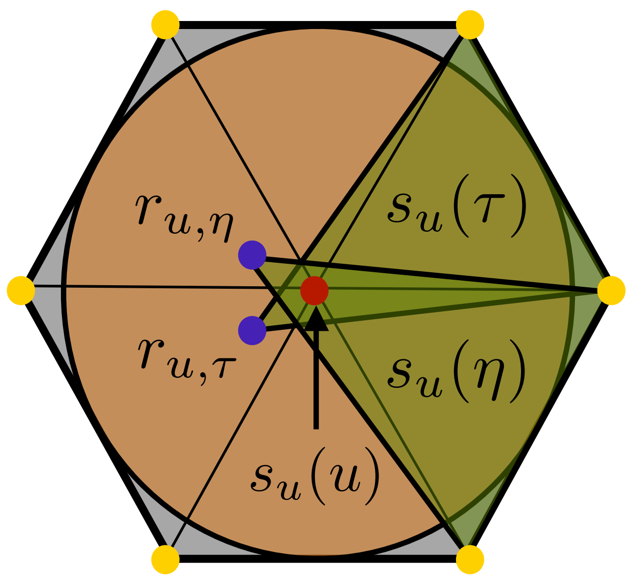

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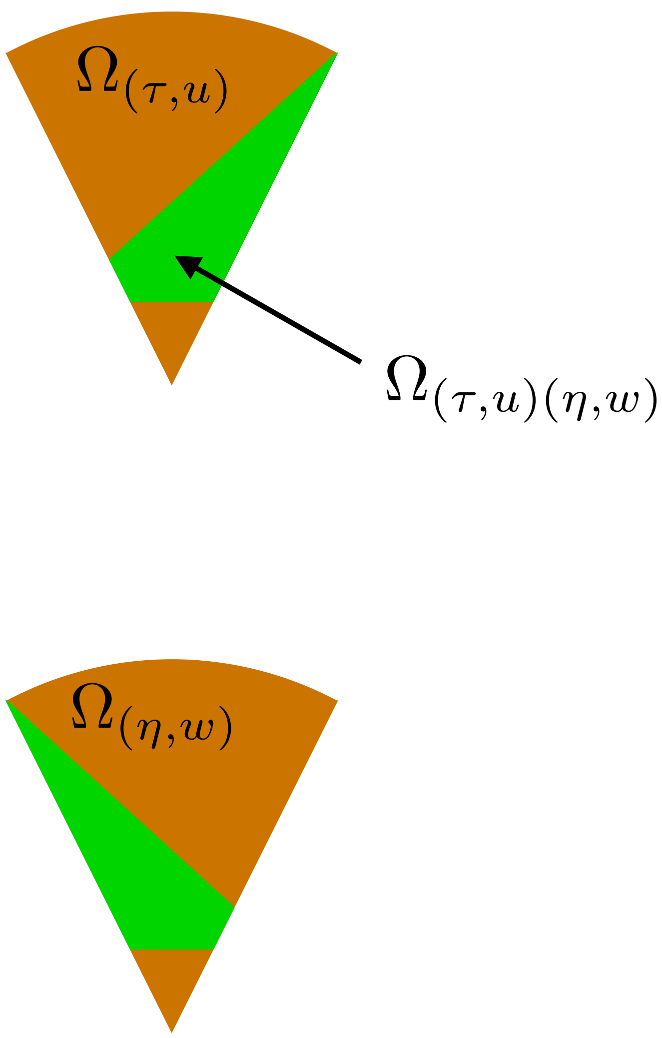

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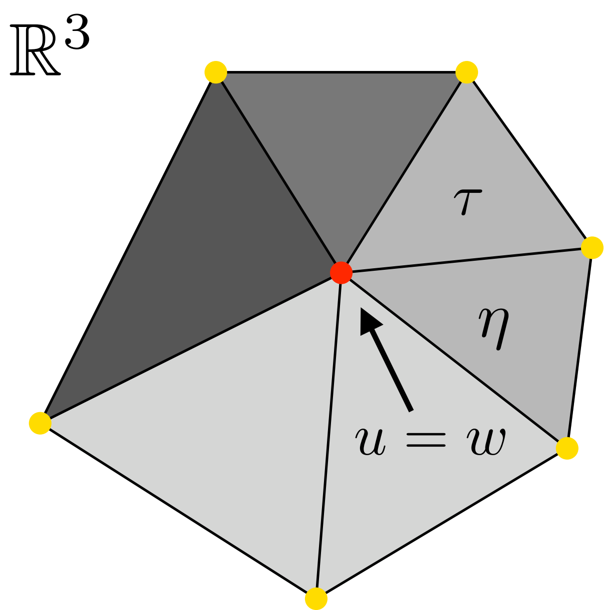

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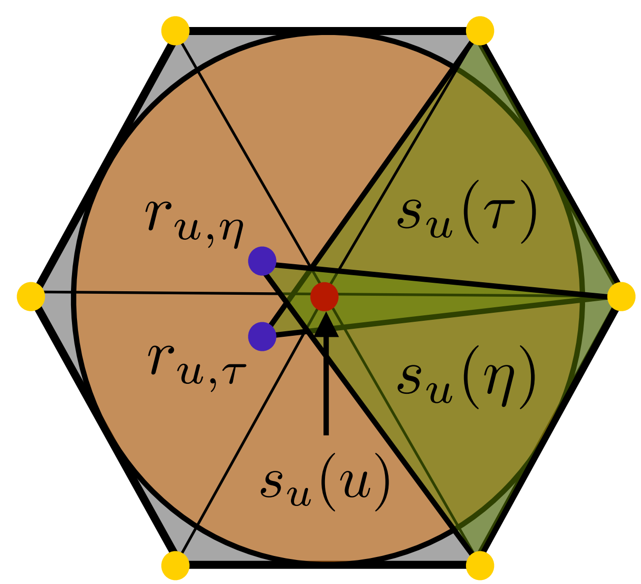

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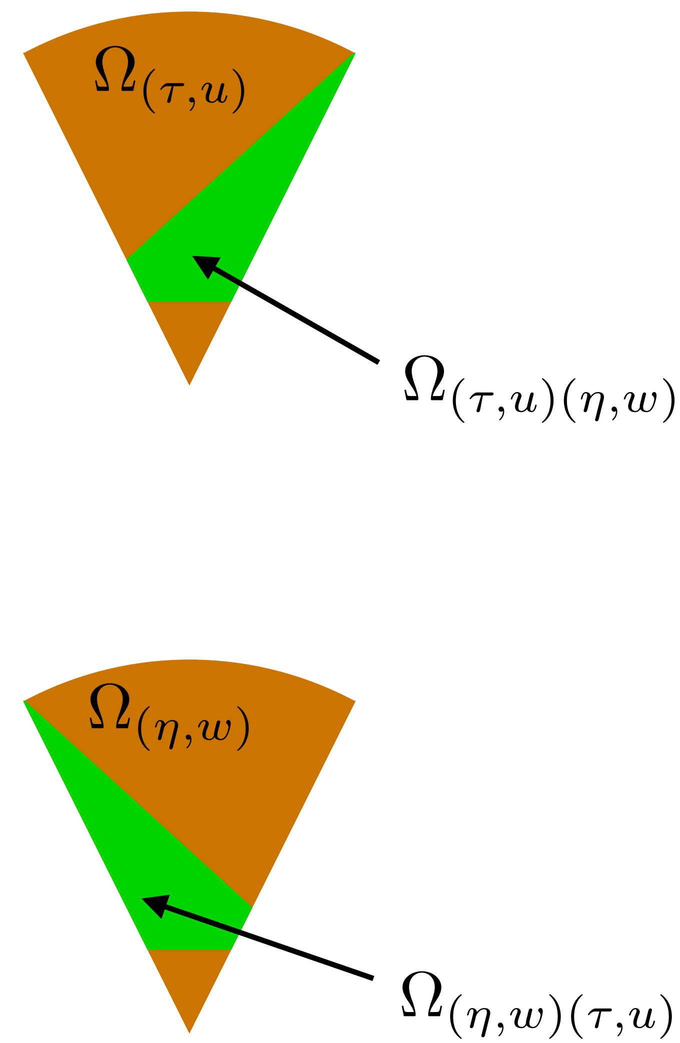

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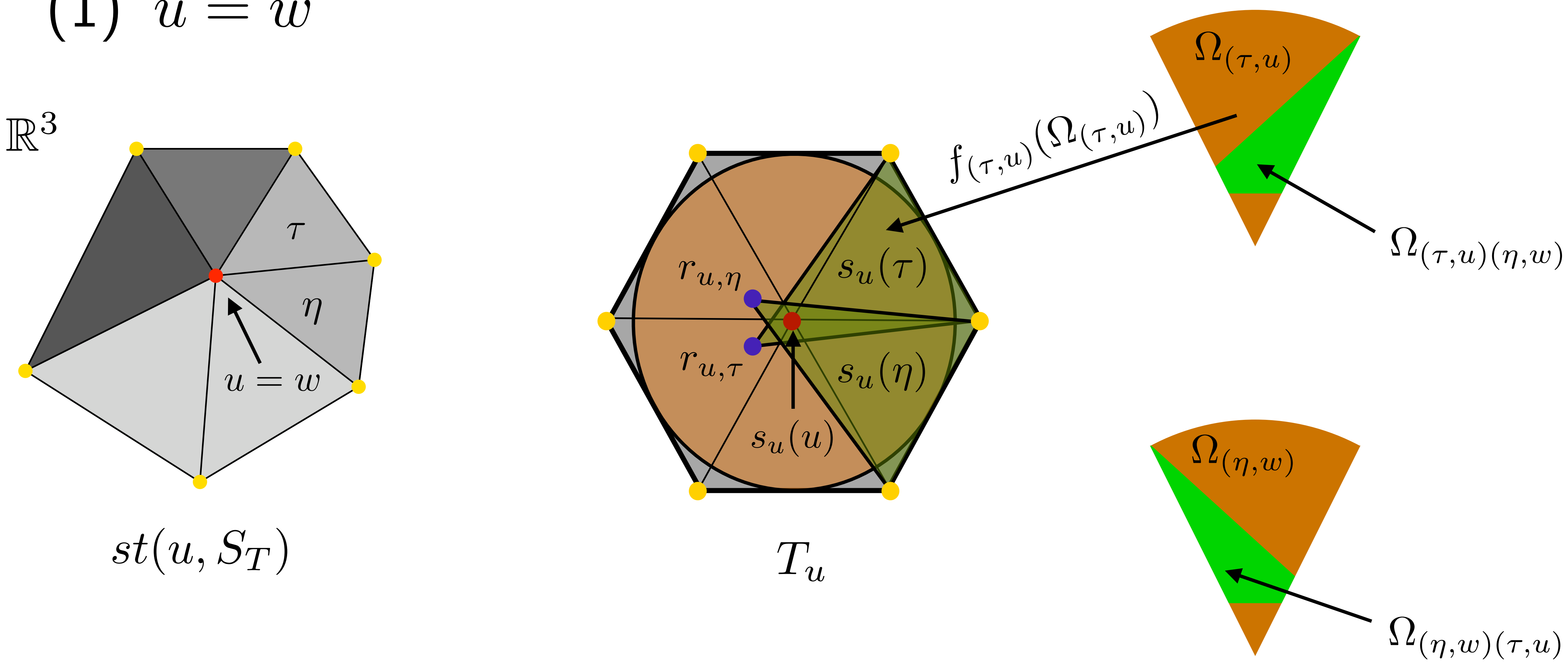

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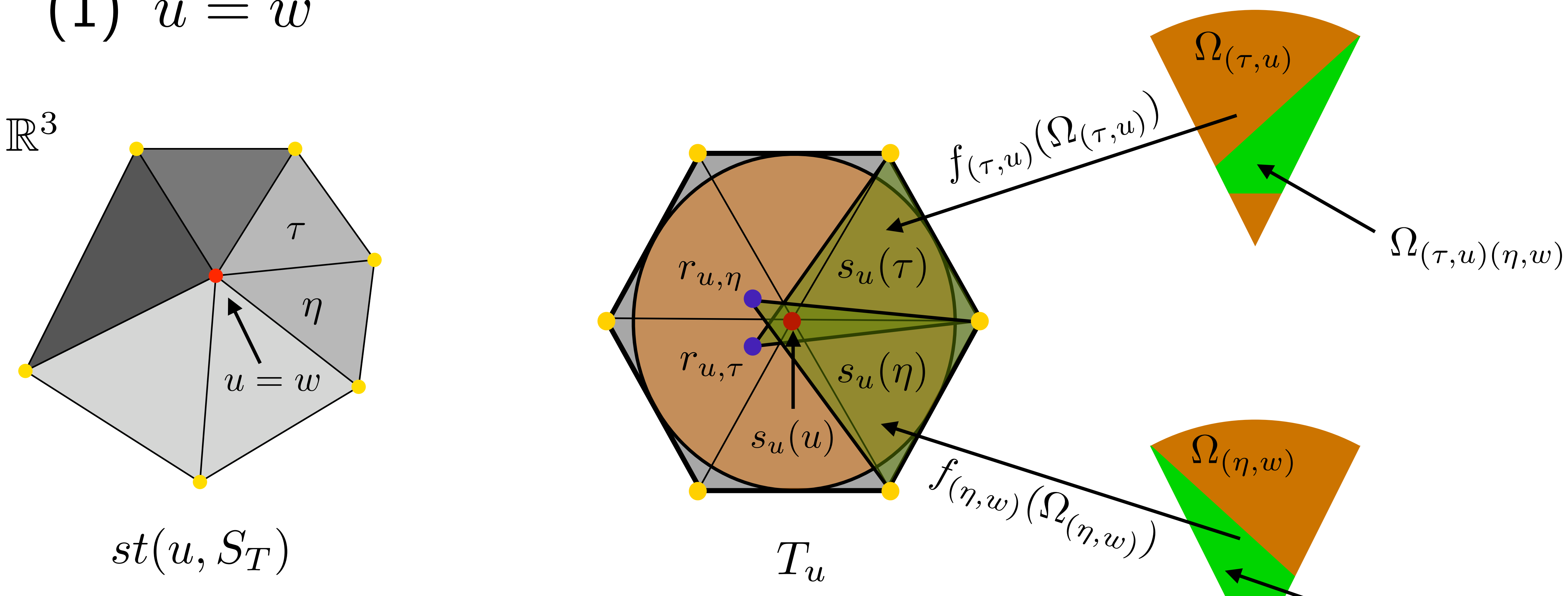

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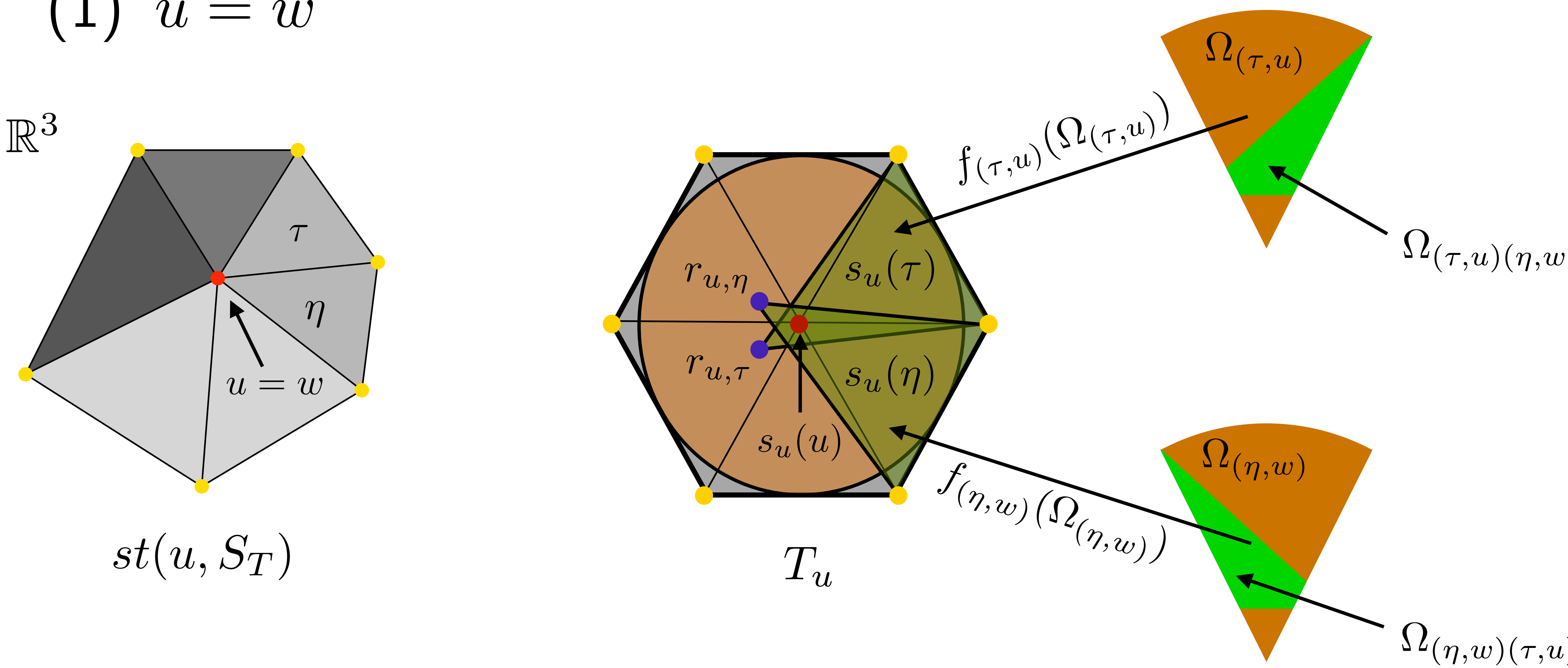

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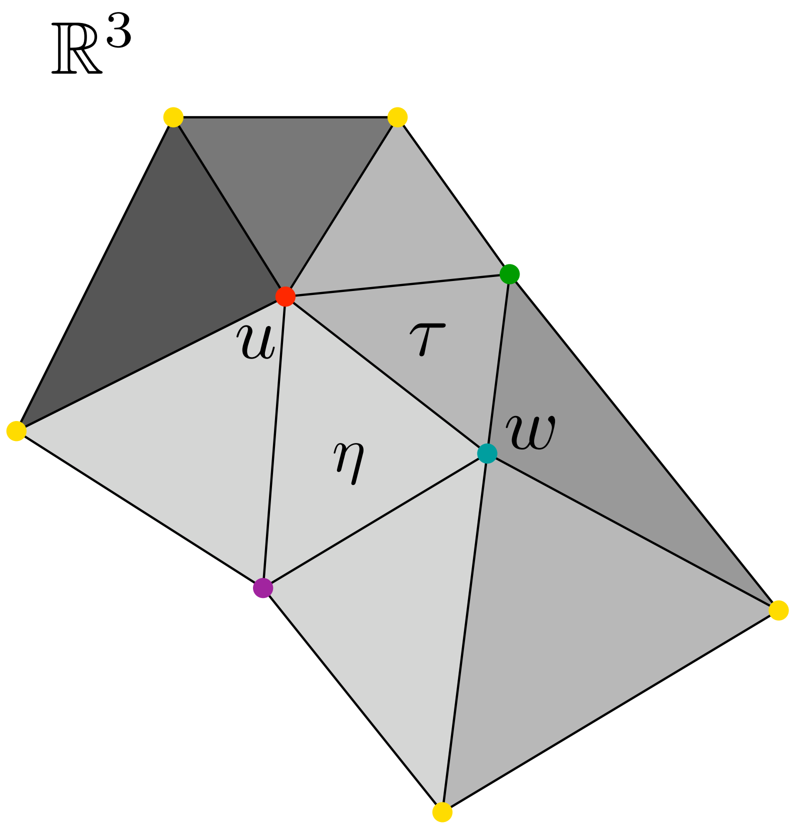

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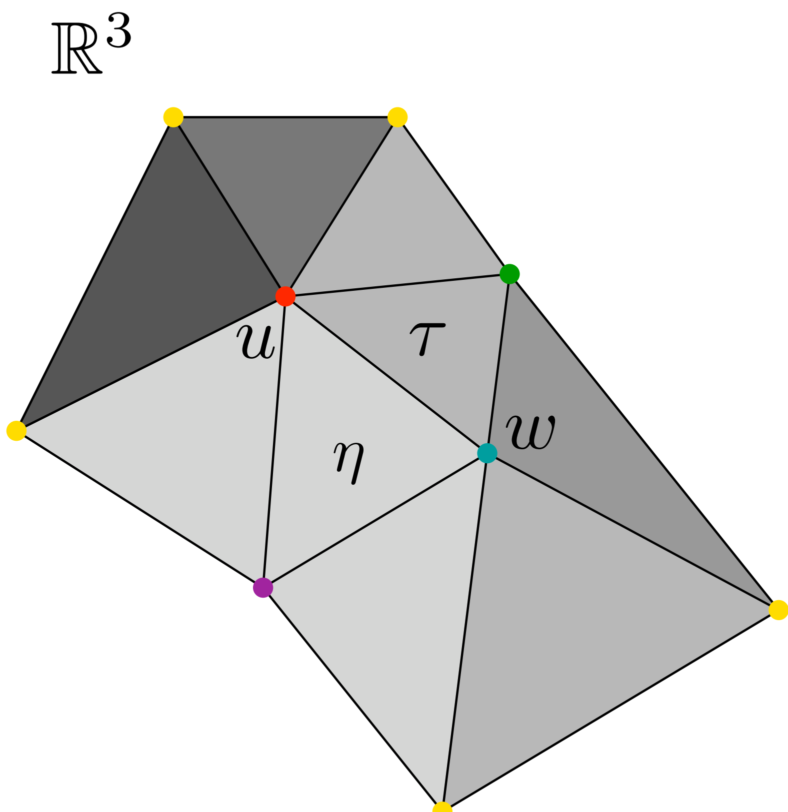

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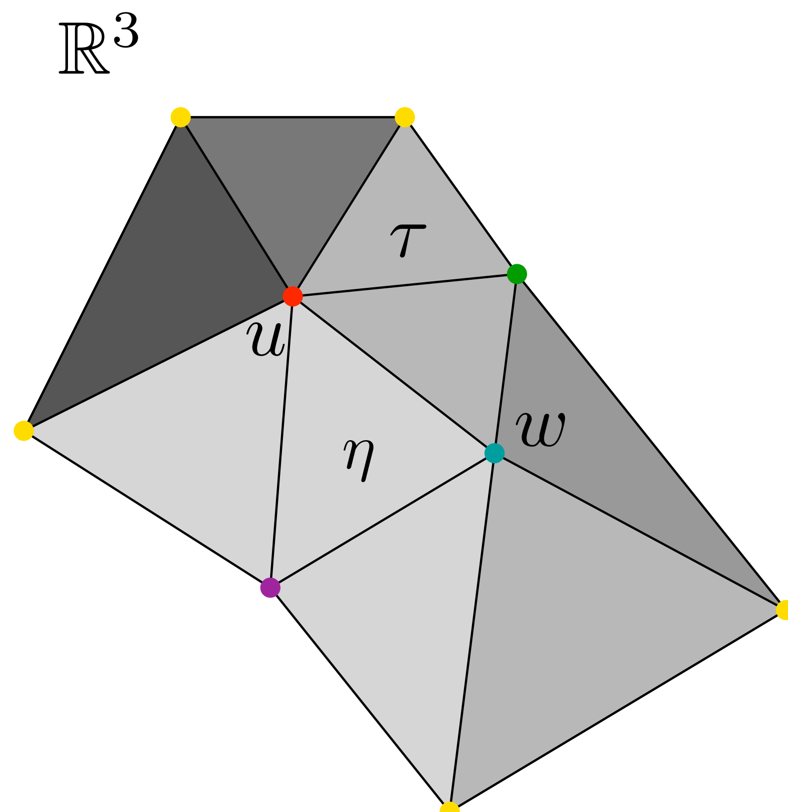

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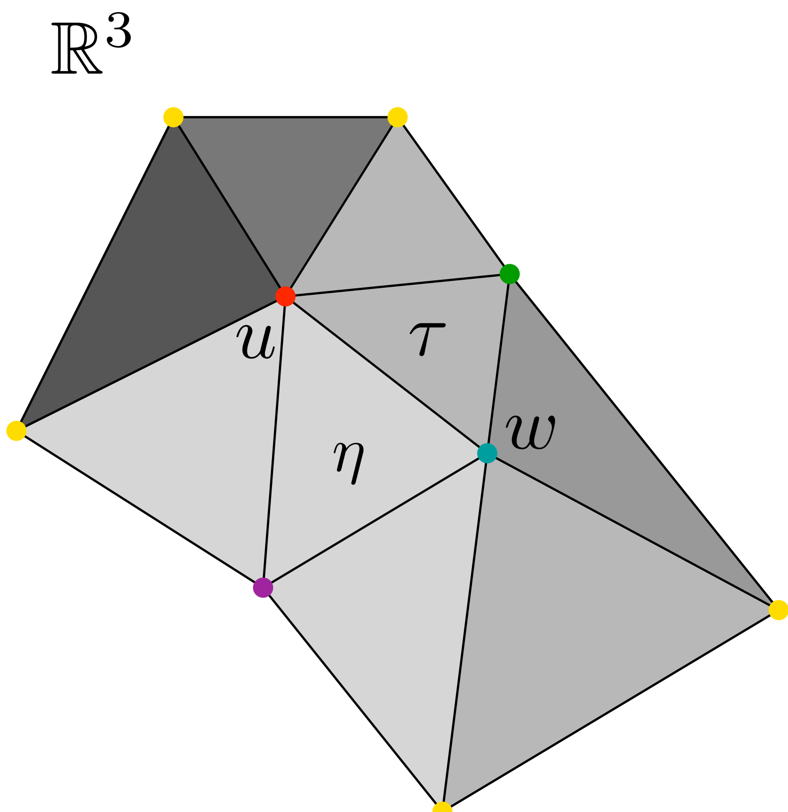

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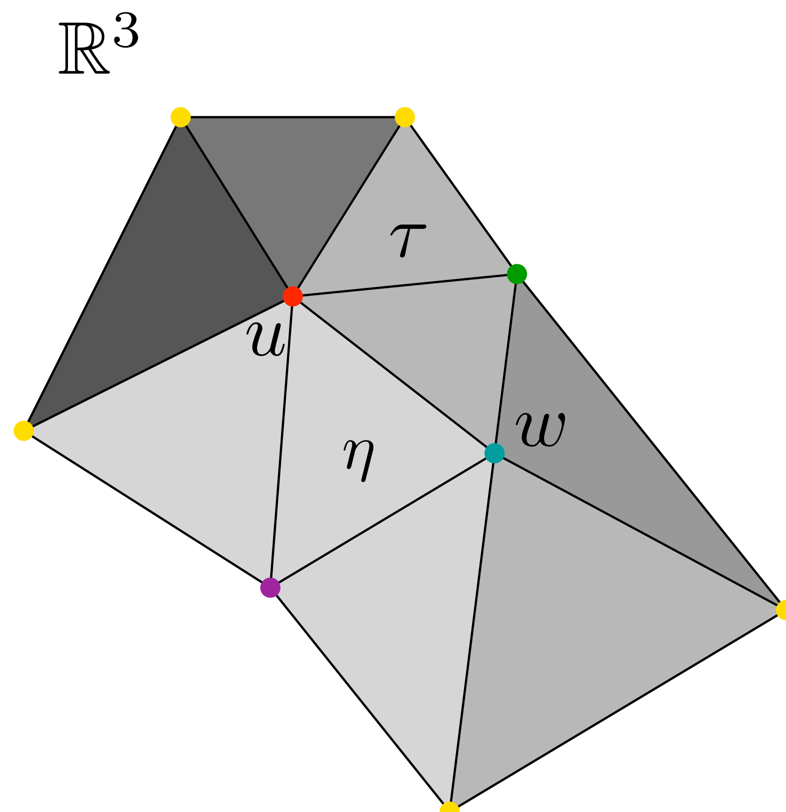

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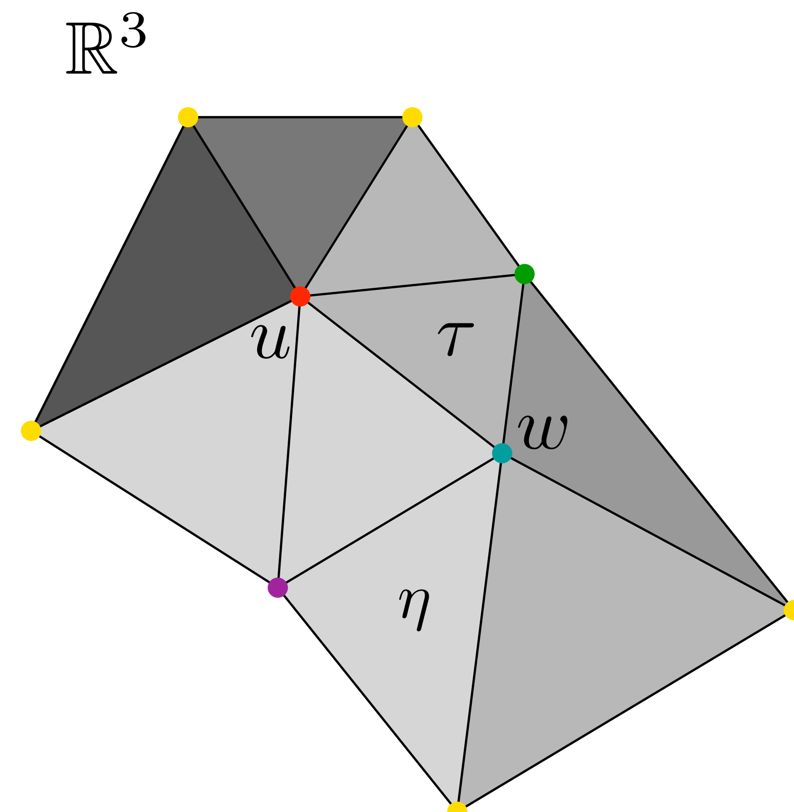

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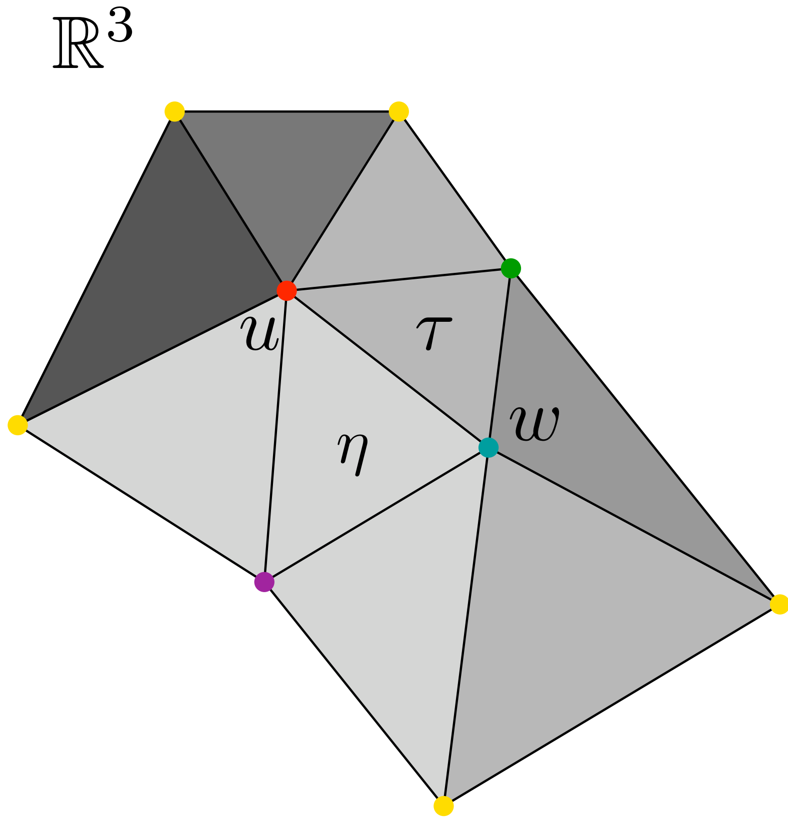

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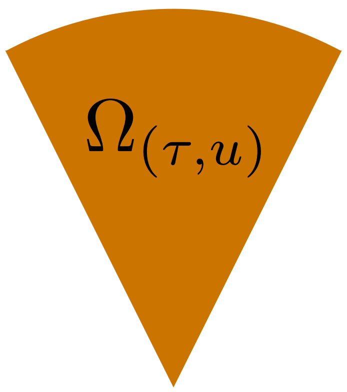

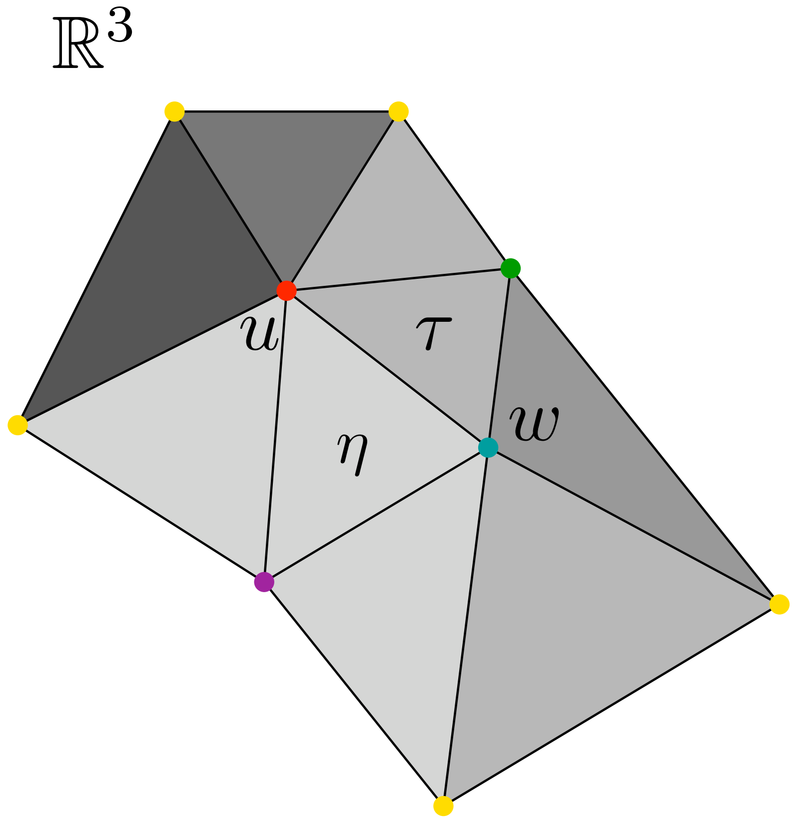

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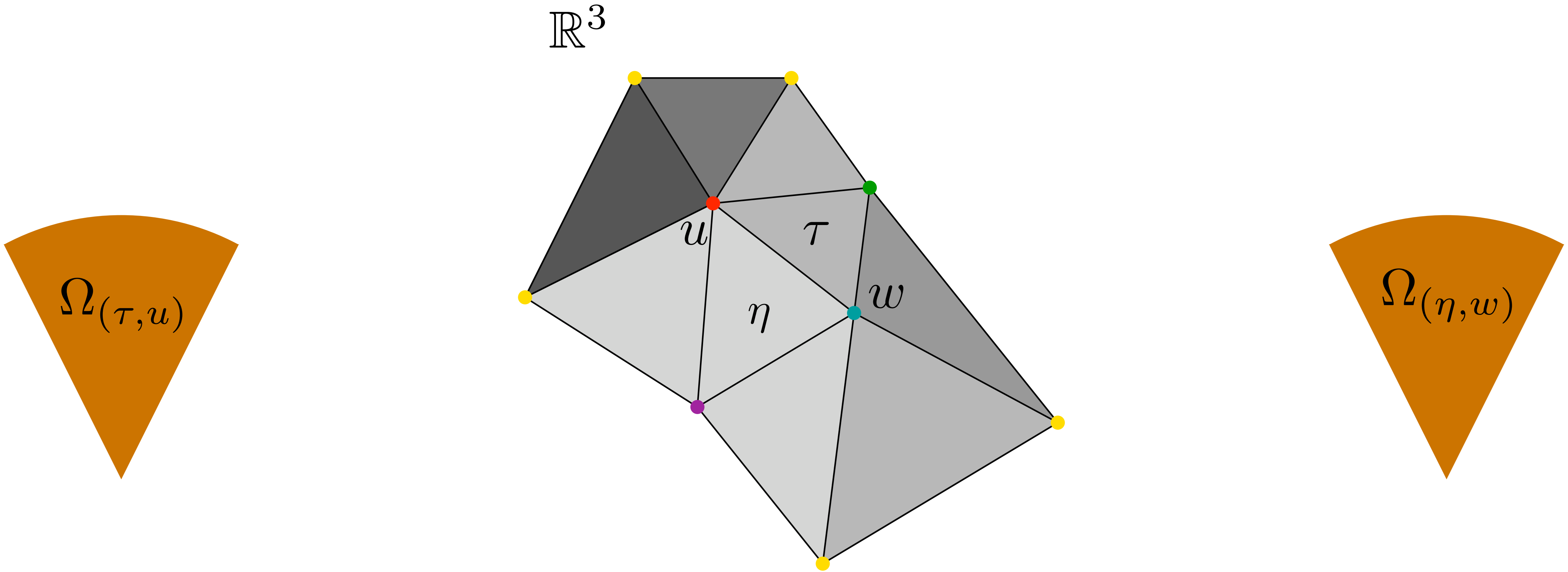

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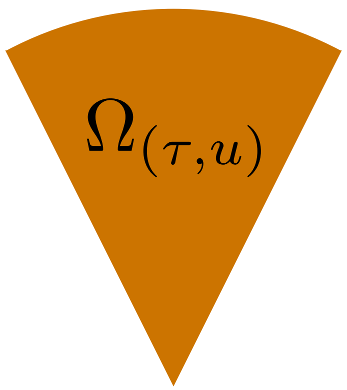

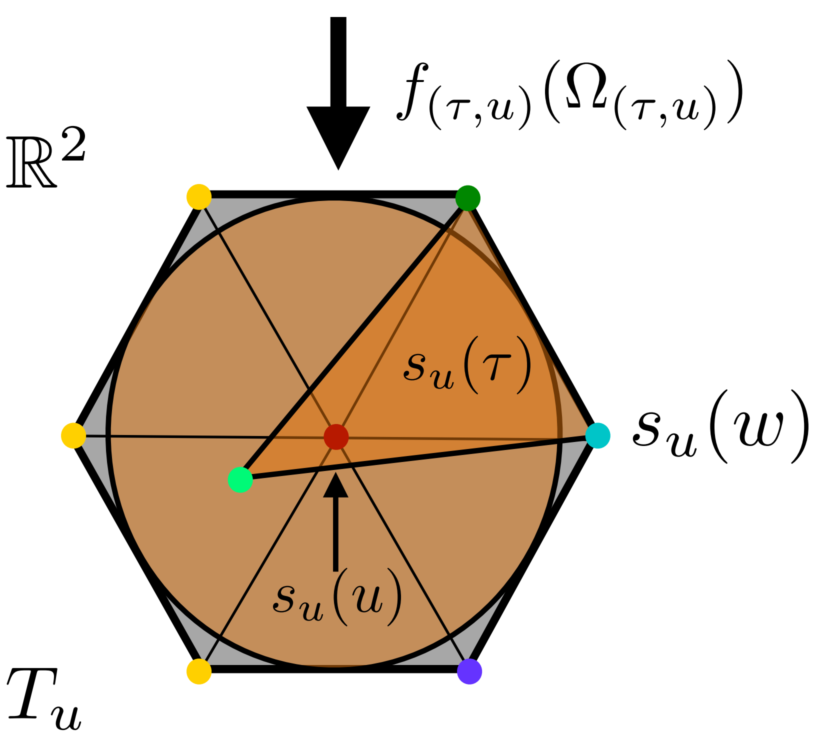

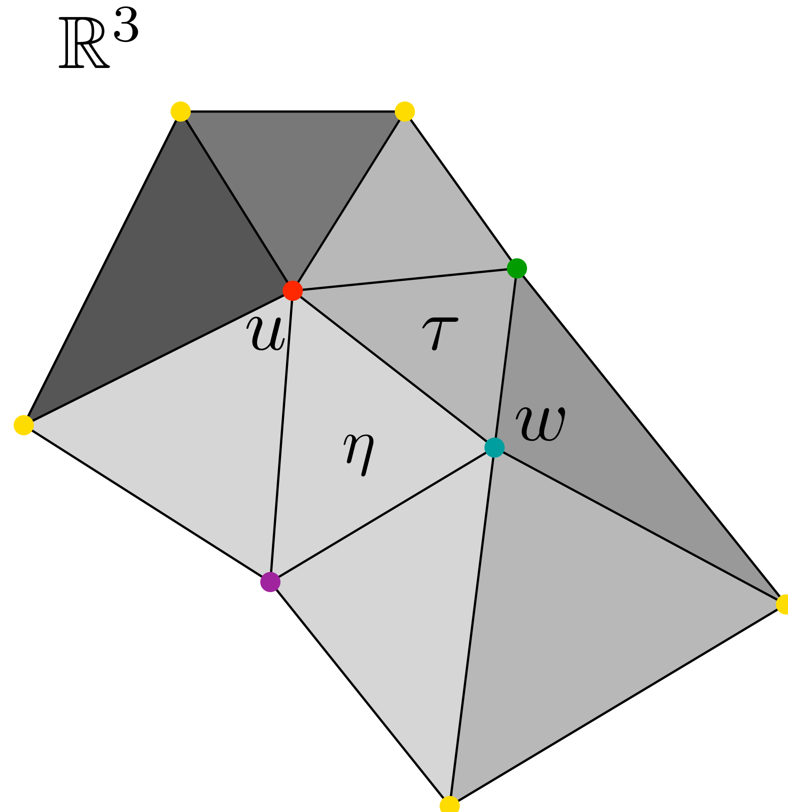

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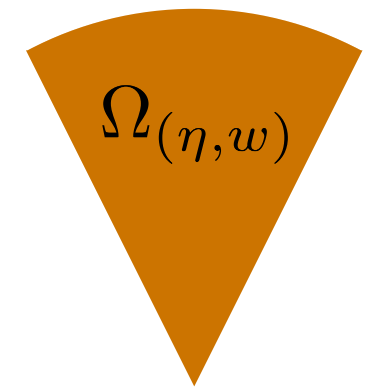

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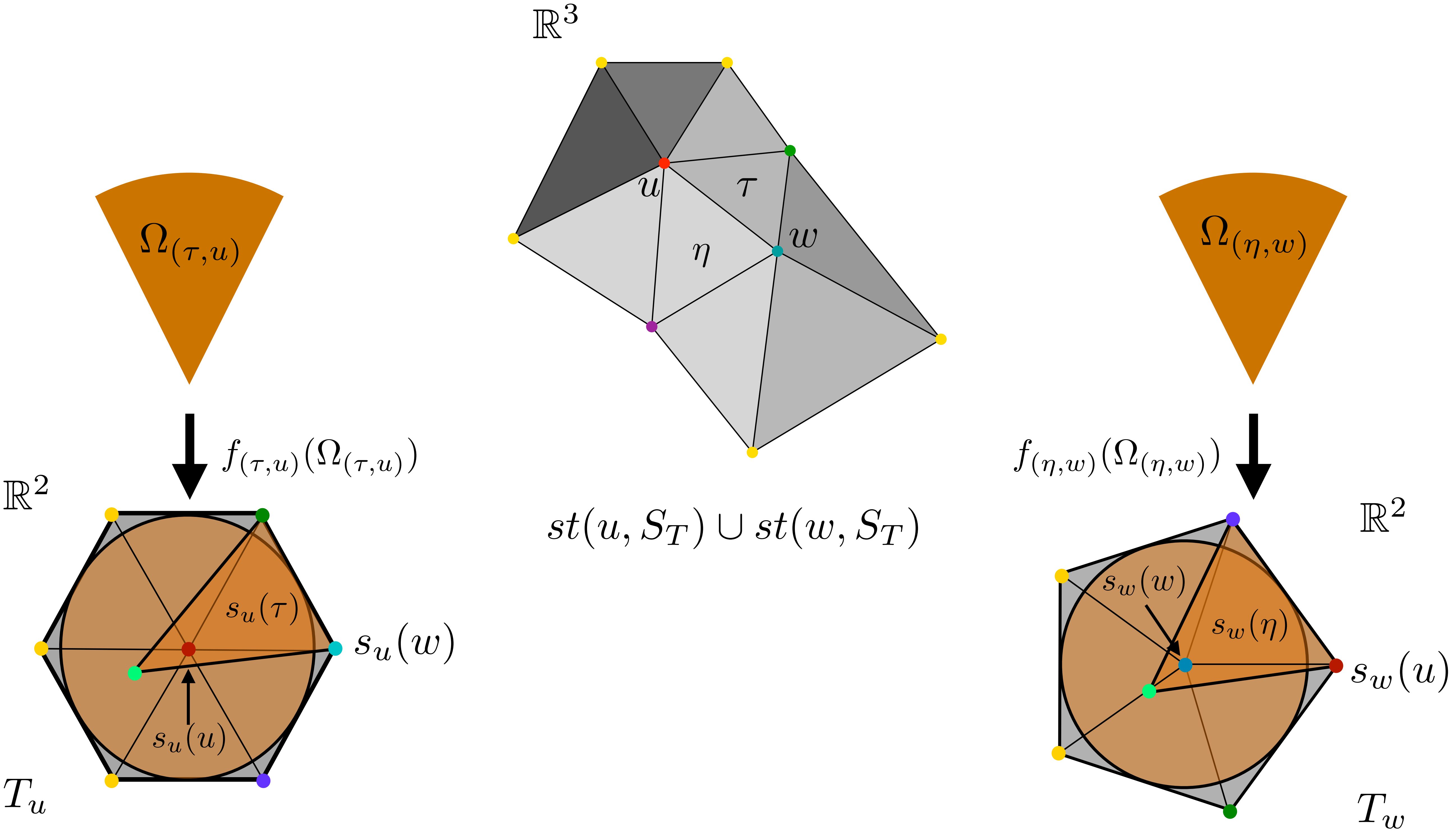

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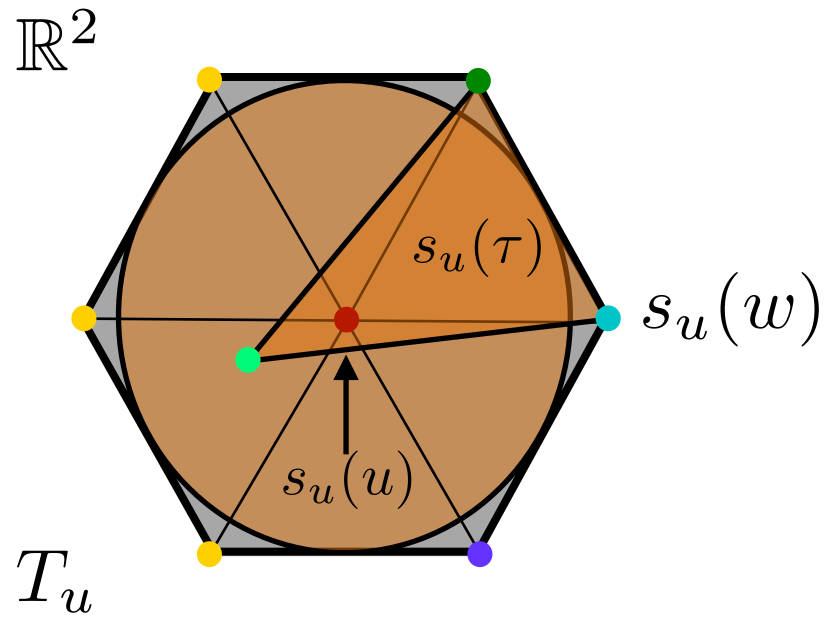

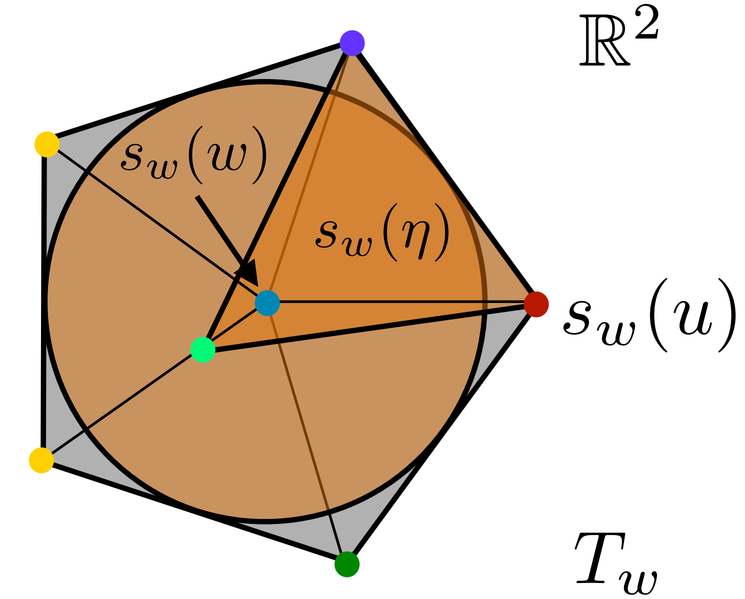

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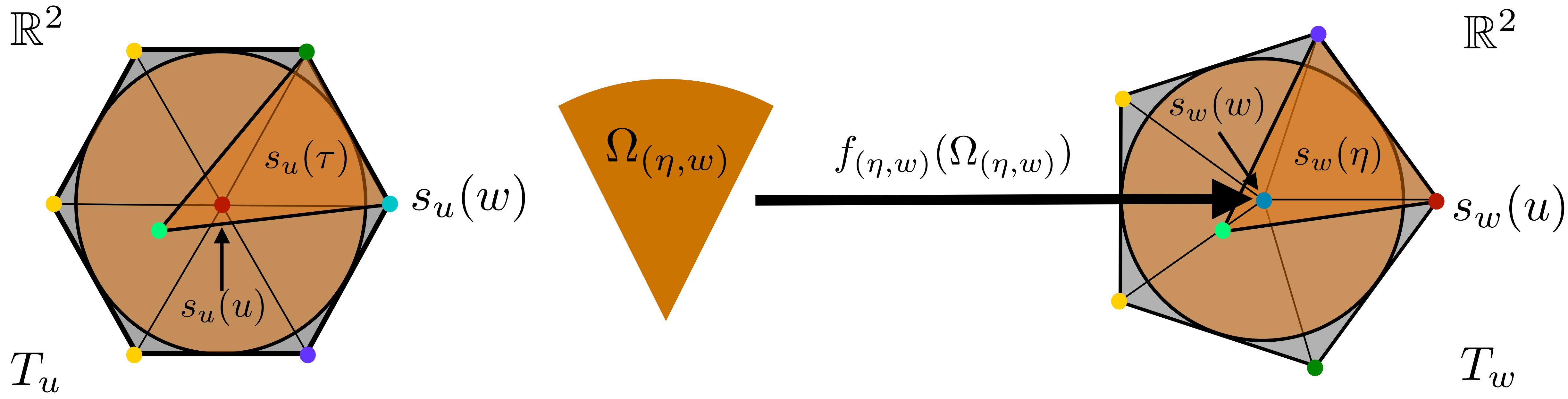

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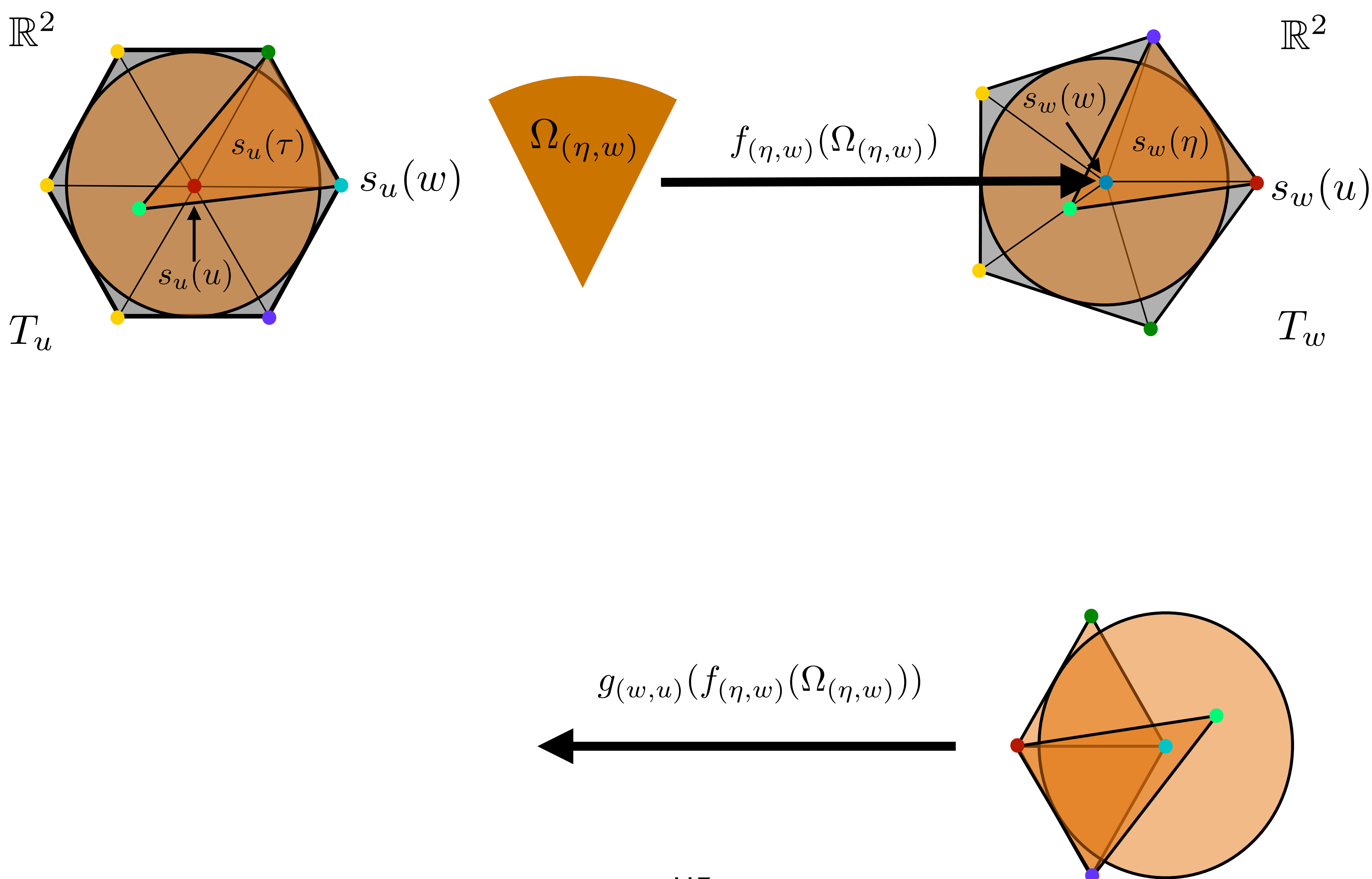

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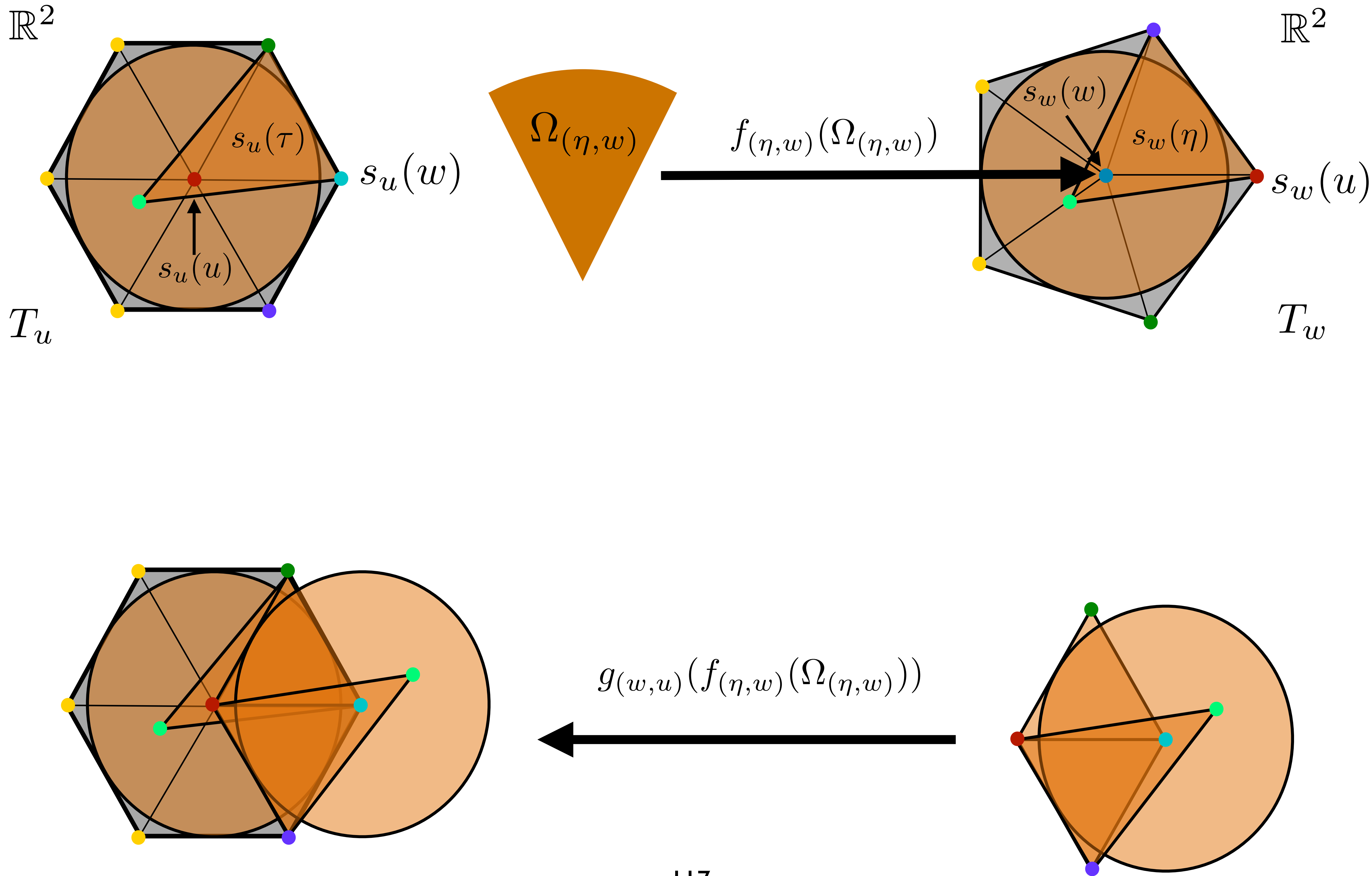

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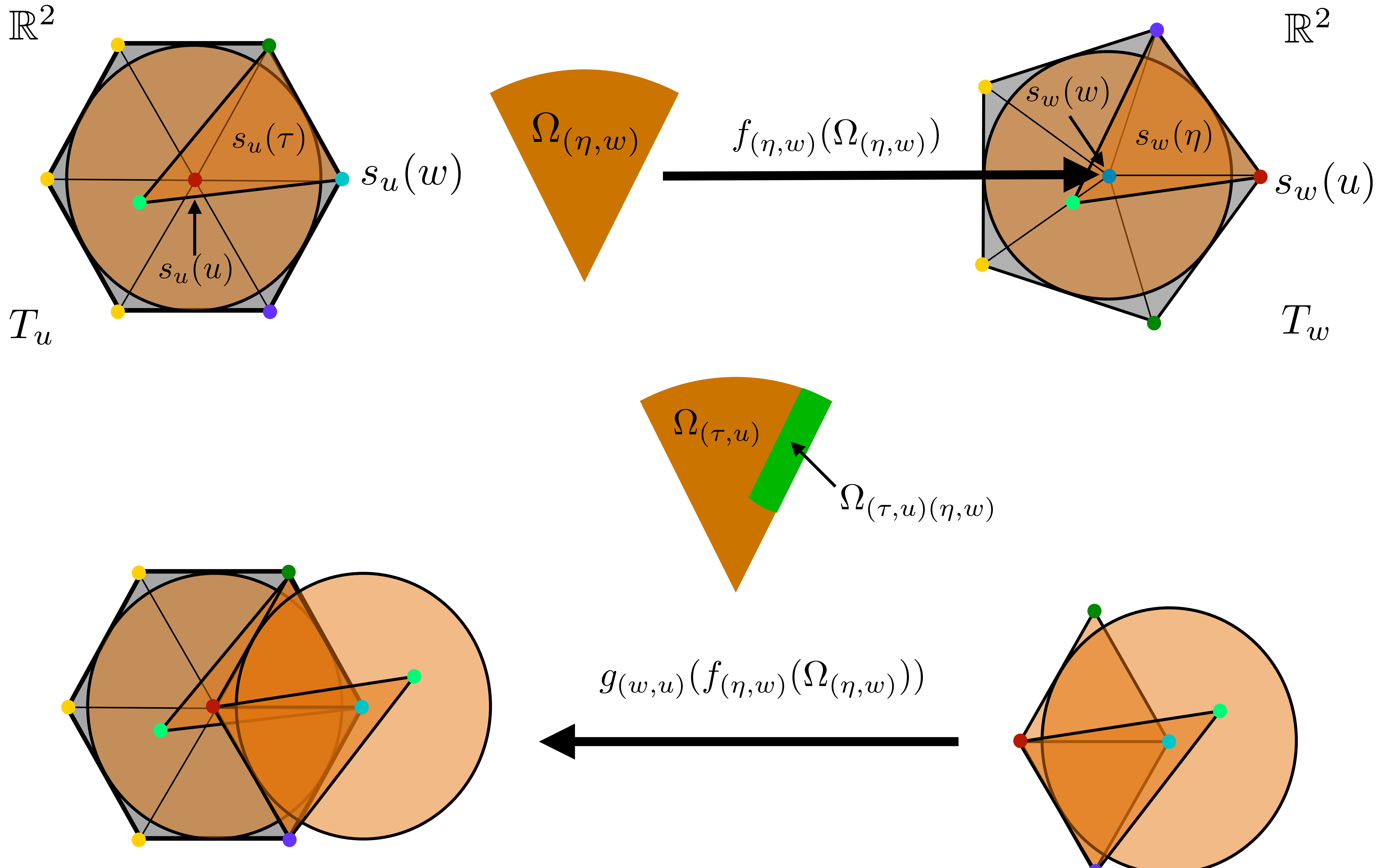

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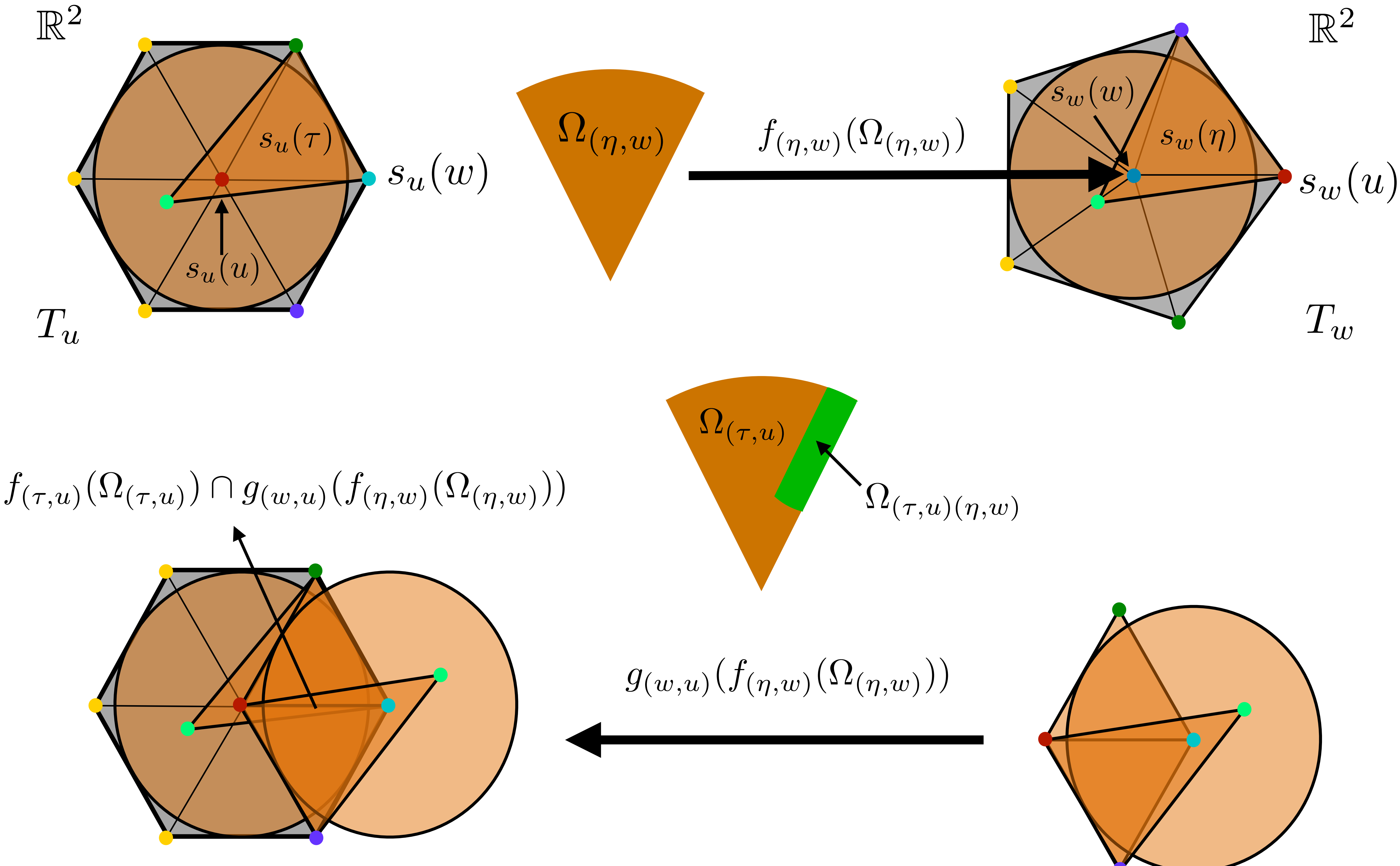

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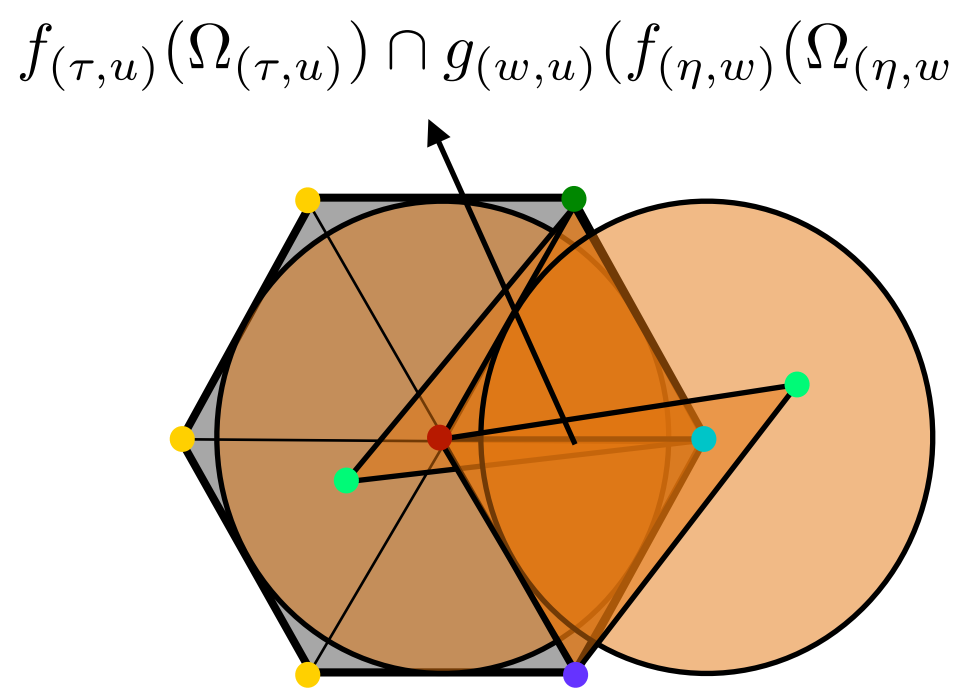

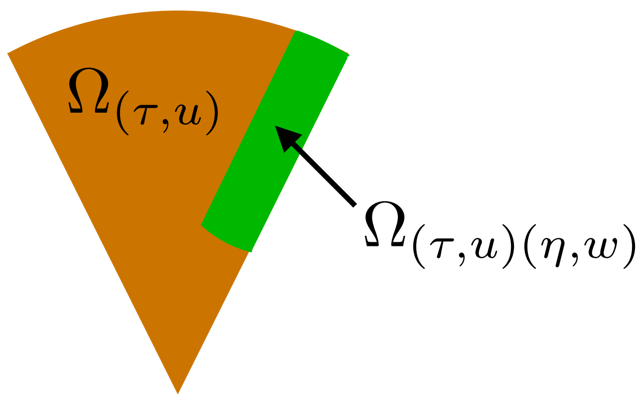

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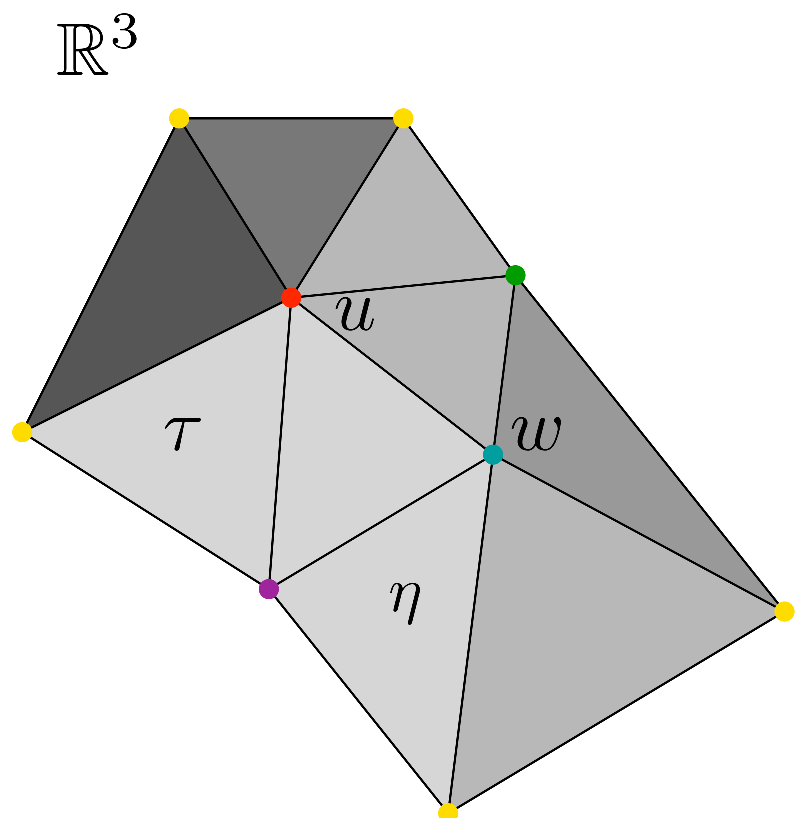

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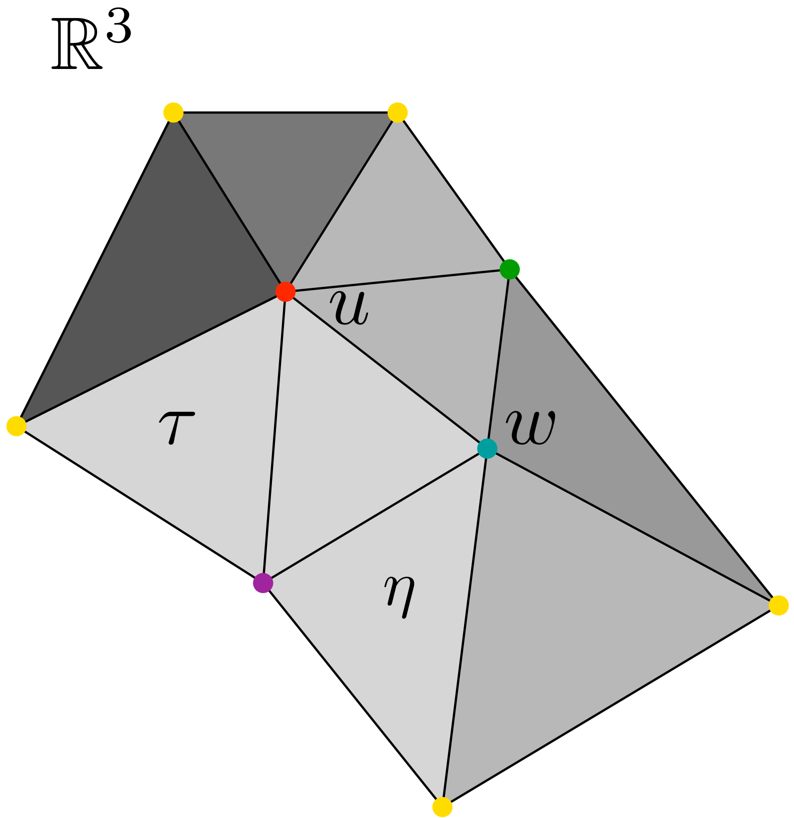

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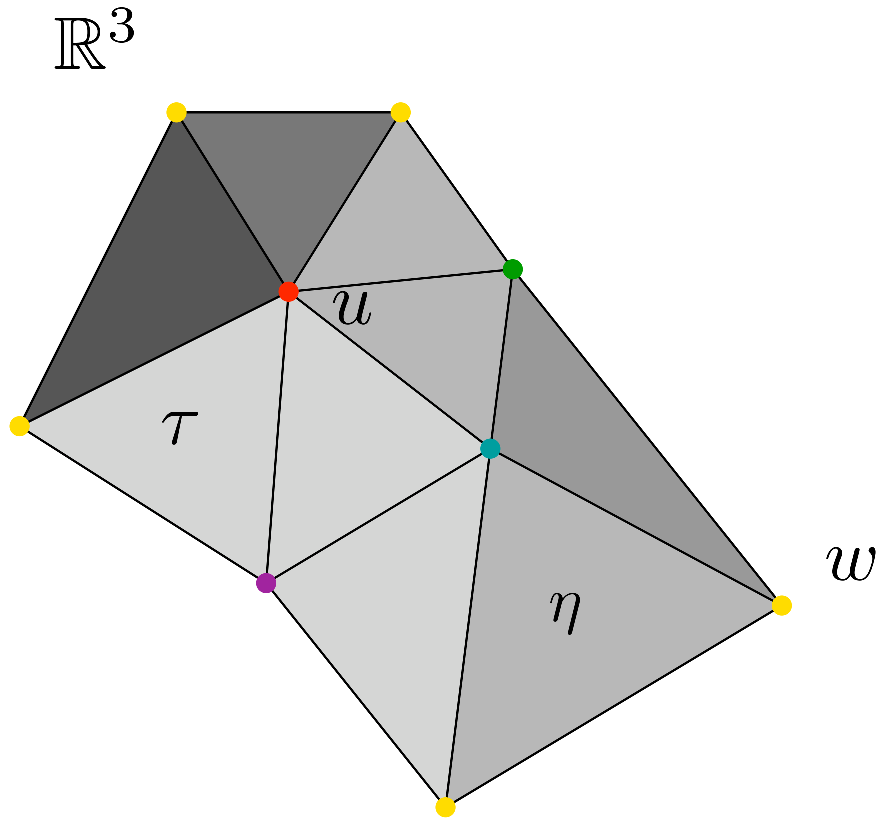

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(2) For every pair $(i, j) \in I \times I$, the set Ω_{ij} is an open subset of Ω_i. Furthermore, $\Omega_{ii} = \Omega_i$ and $\Omega_{ji} \neq \emptyset$ if and only if $\Omega_{ij} \neq \emptyset$.

Fitting Surfaces to Polygonal Meshes (Part II)

Marcelo Siqueira
UFMS

Outline

- Building a Set of Gluing Data
- The User's Perspective
- Building Parametrizations
- Results
- Conclusions

Building a Set of Gluing Data

Building a Set of Gluing Data

Transition functions

Building a Set of Gluing Data

Transition functions

Let

$$K = \{((\tau, u), (\eta, w)) \in I \times I \mid \Omega_{(\tau,u),(\eta,w)} \neq \emptyset\}\,.$$

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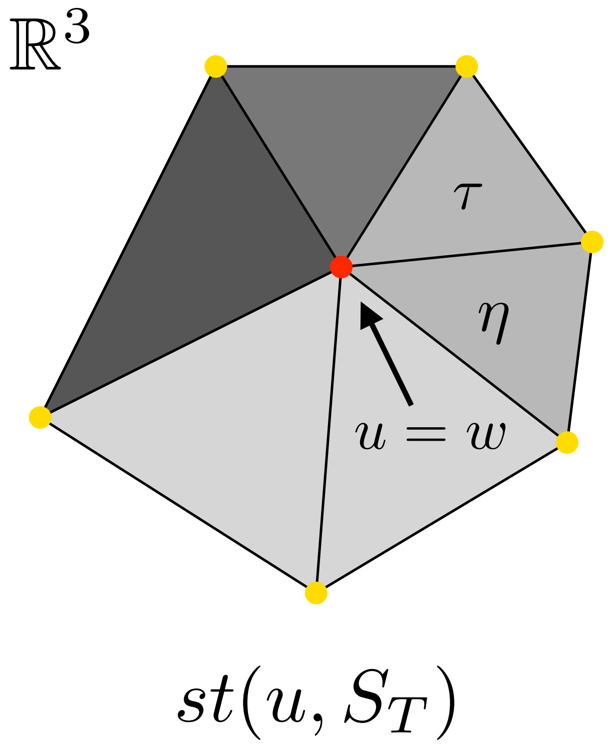

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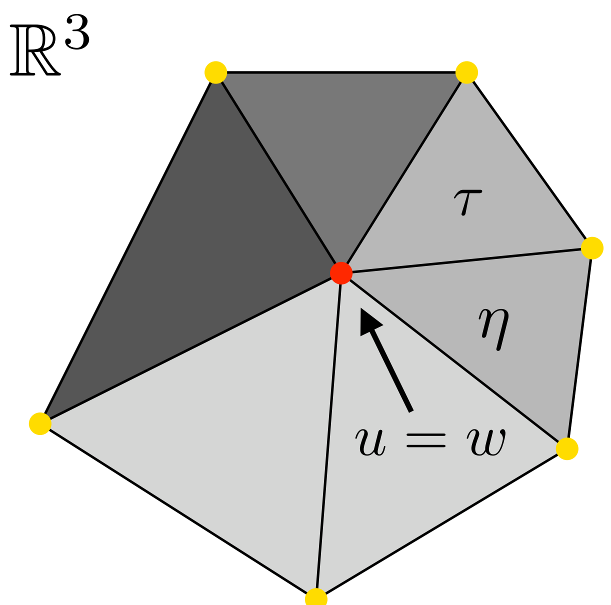

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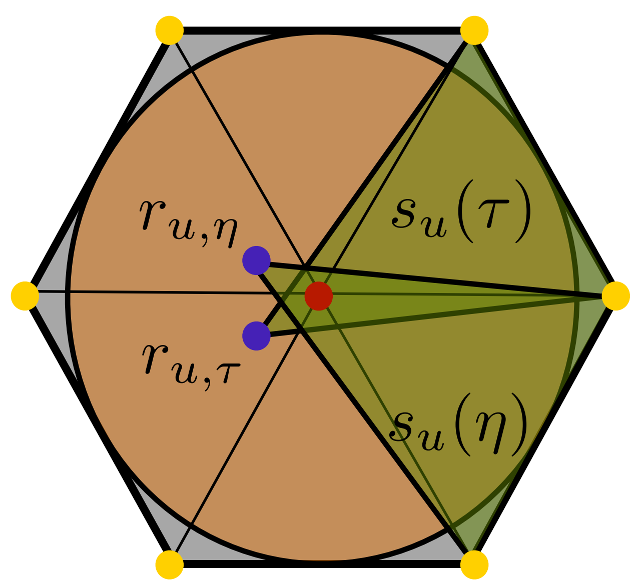

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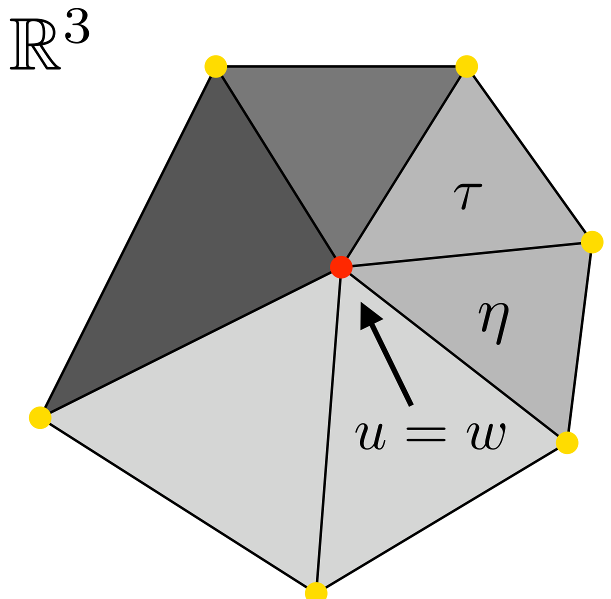

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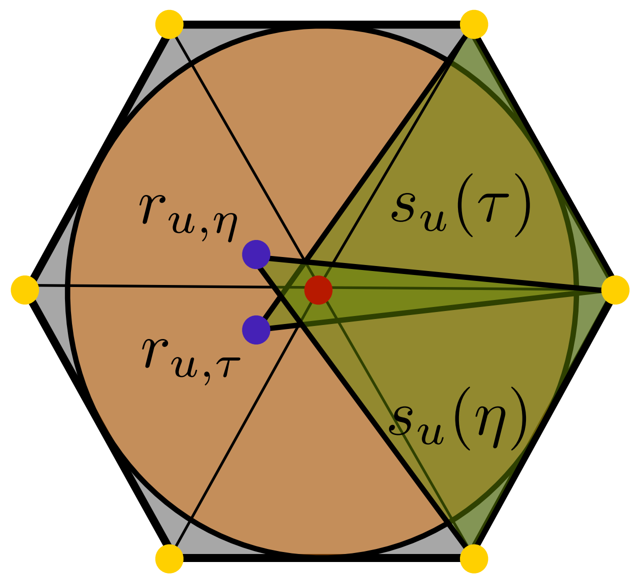

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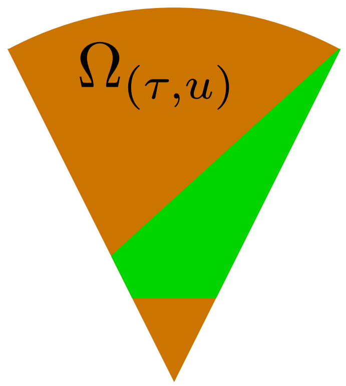

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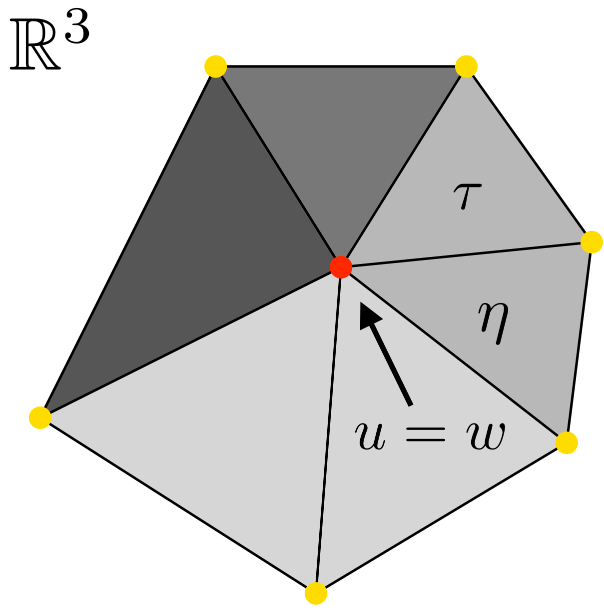

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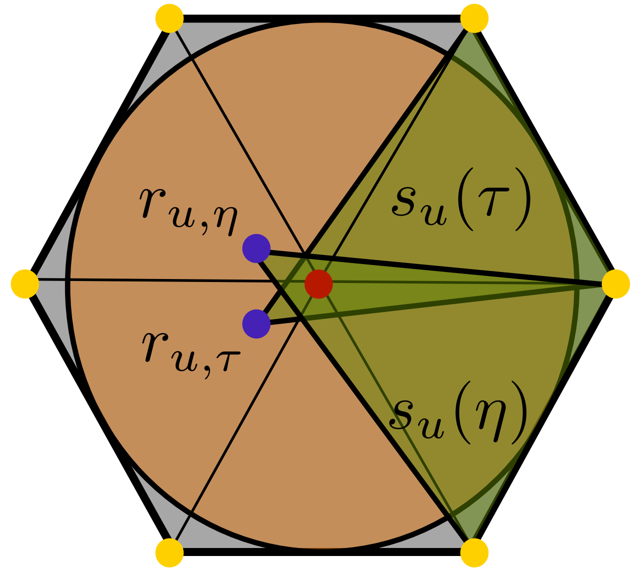
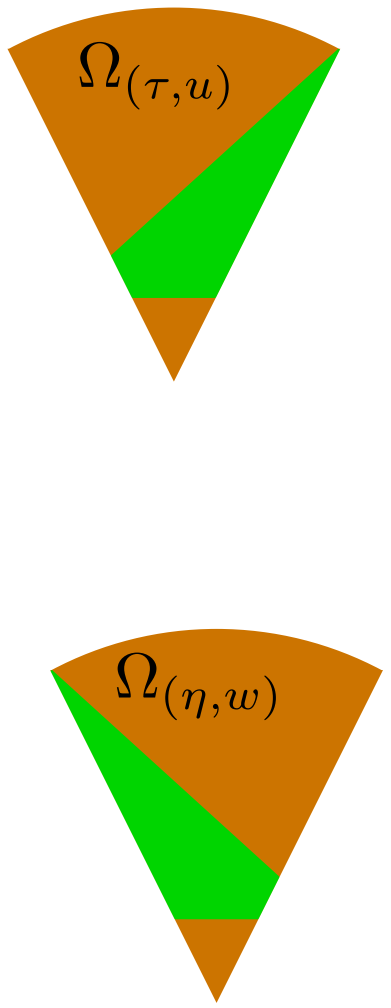
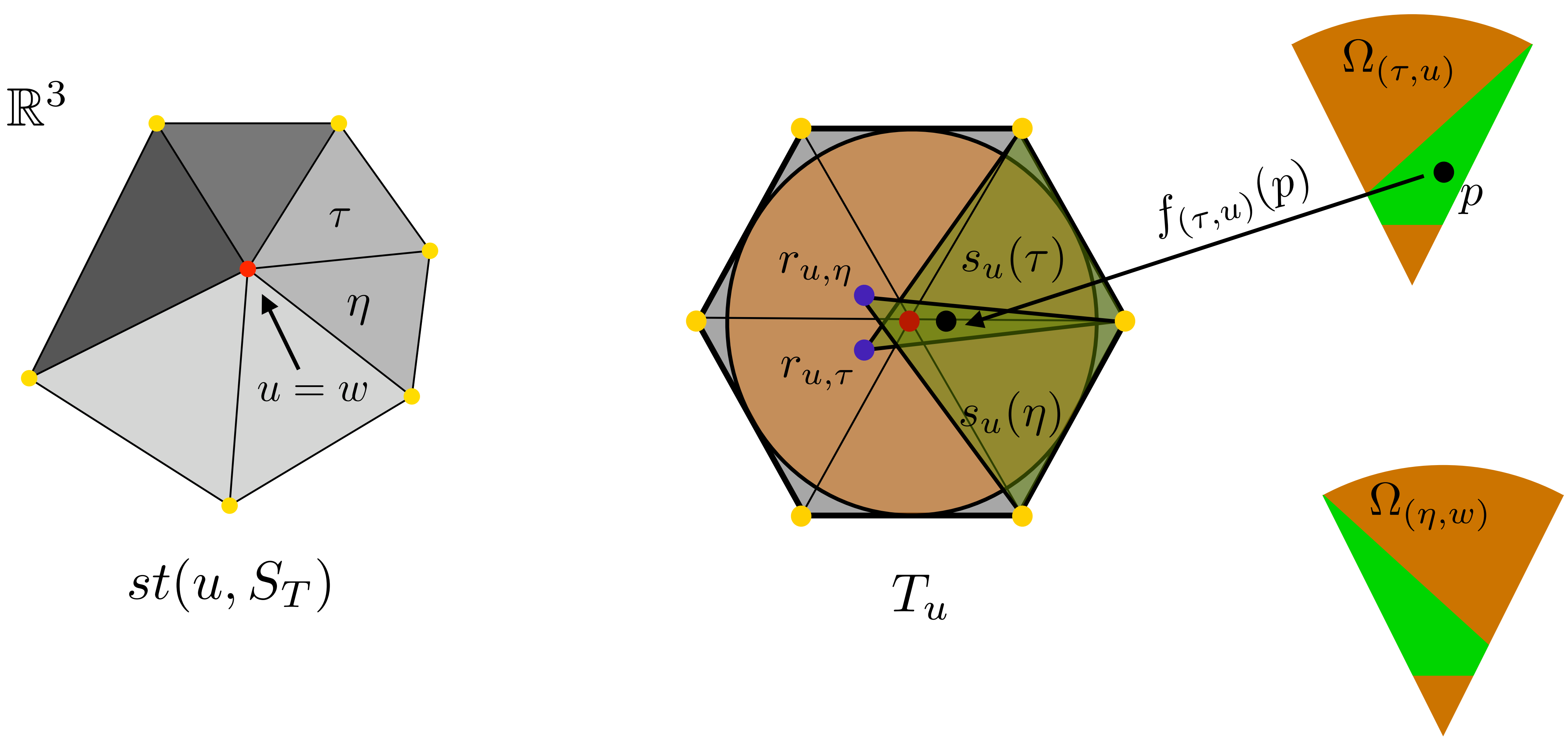


T_u



Building a Set of Gluing Data

(1) $u = w$



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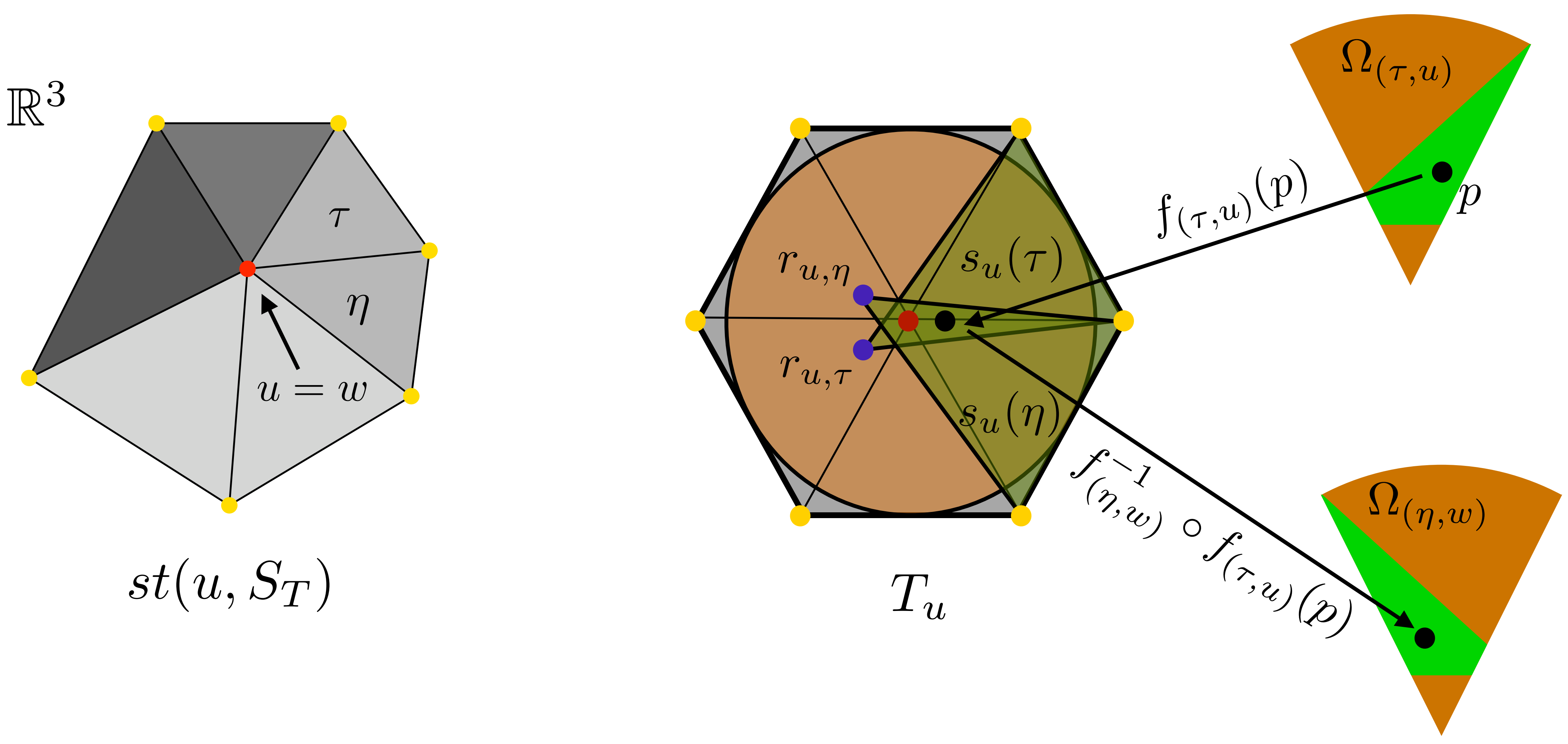

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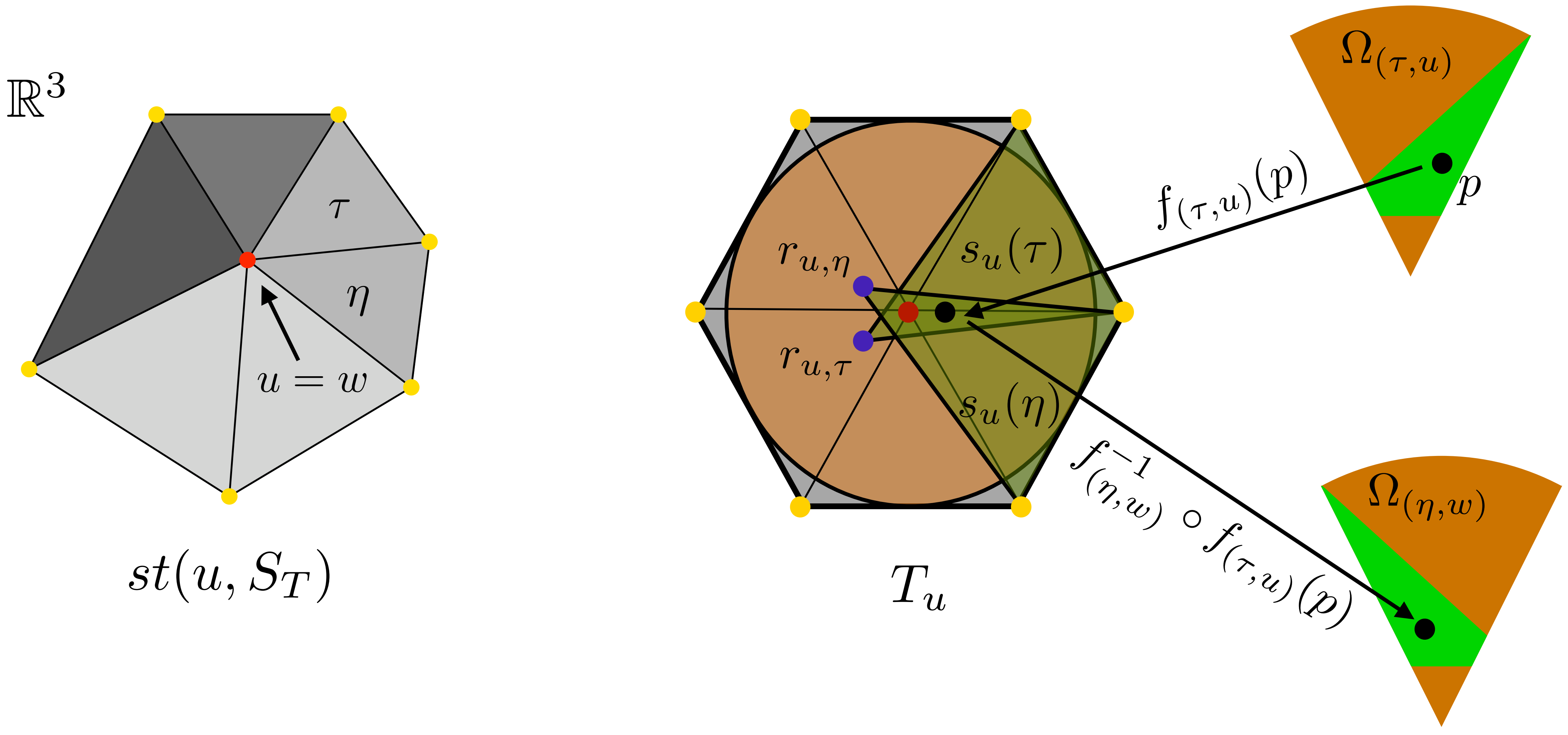
$$f^{-1}_{(\eta,w)} \circ f_{(\tau,u)}(p)$$



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(2) otherwise

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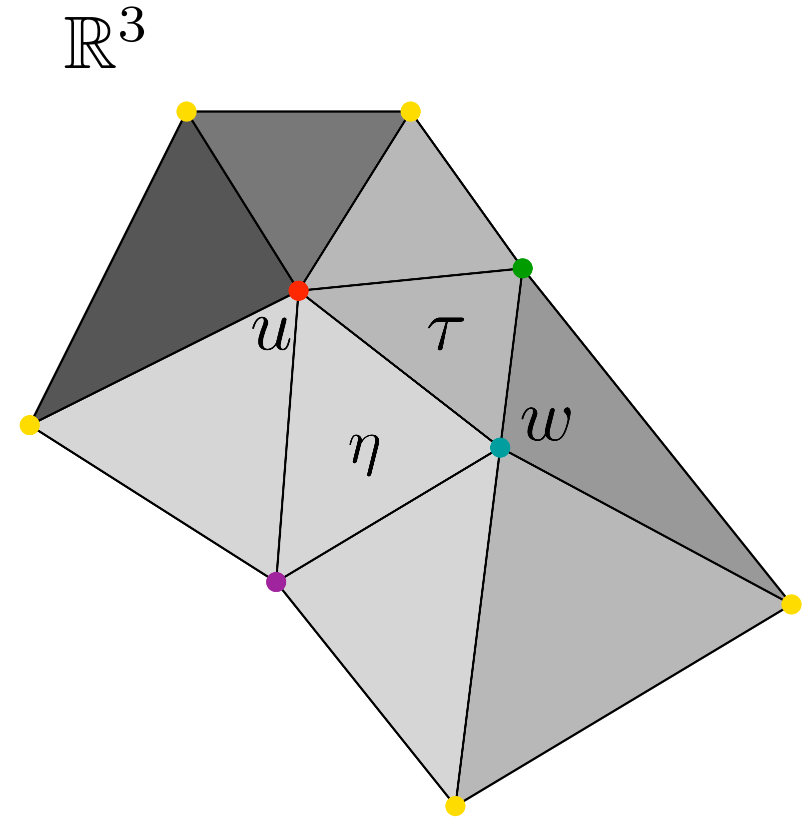

$st(u, S_T) \cup st(w, S_T)$

Building a Set of Gluing Data

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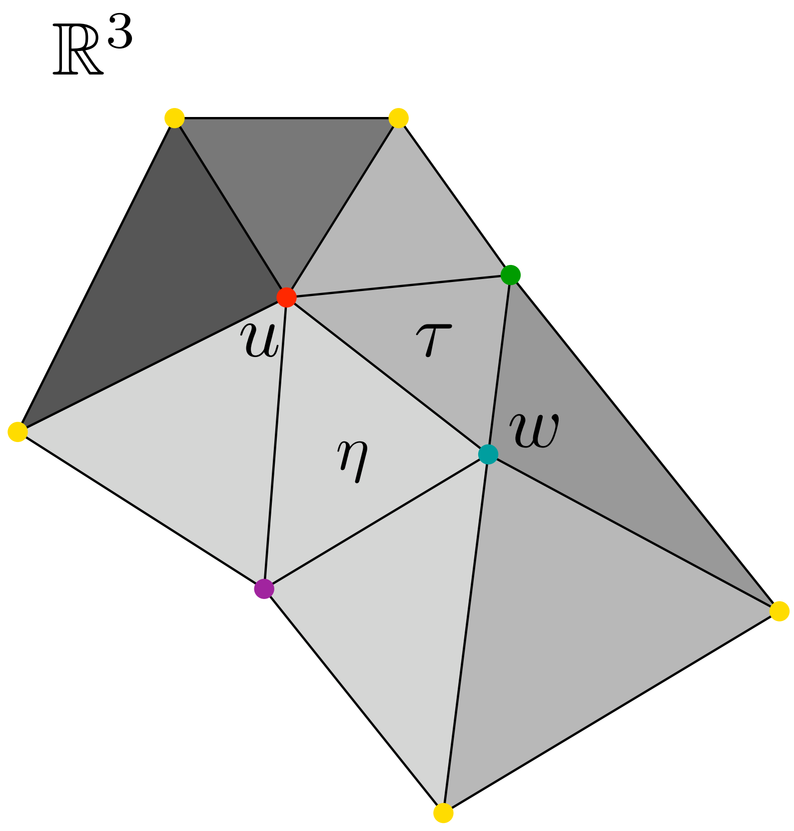

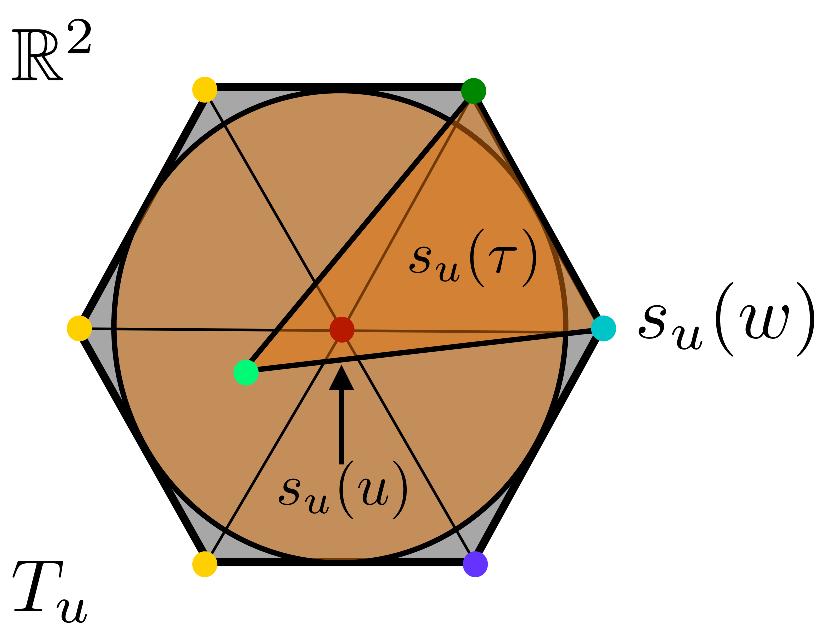

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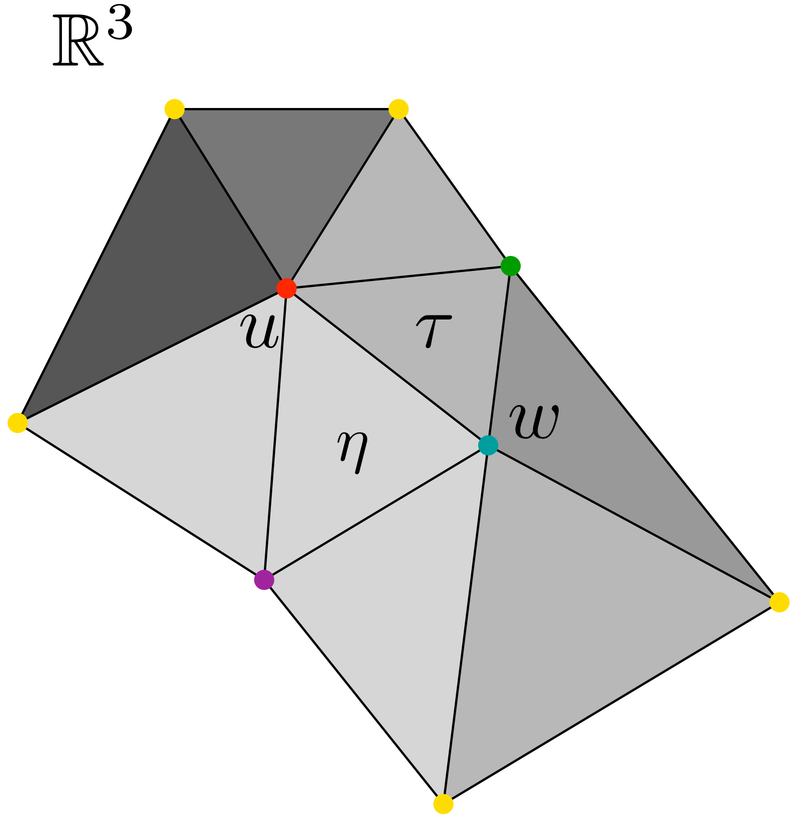

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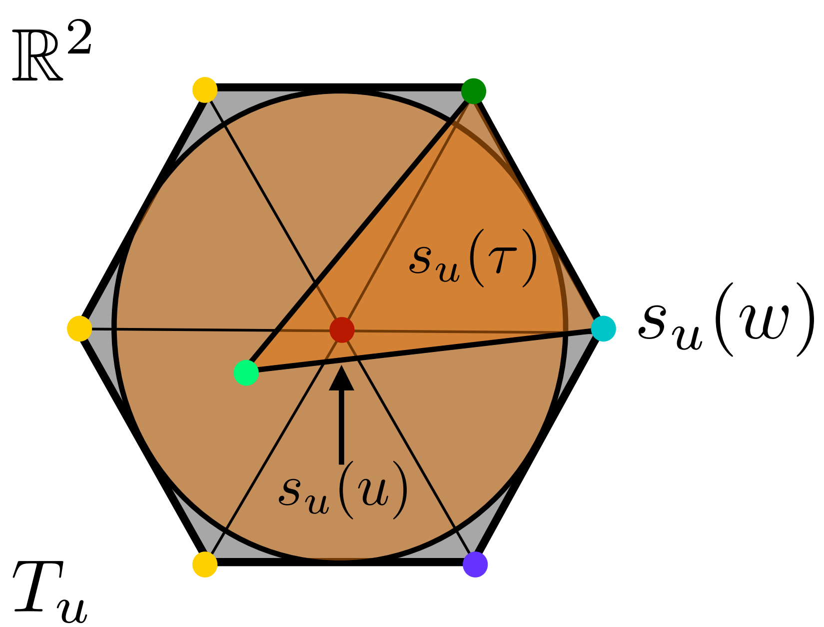

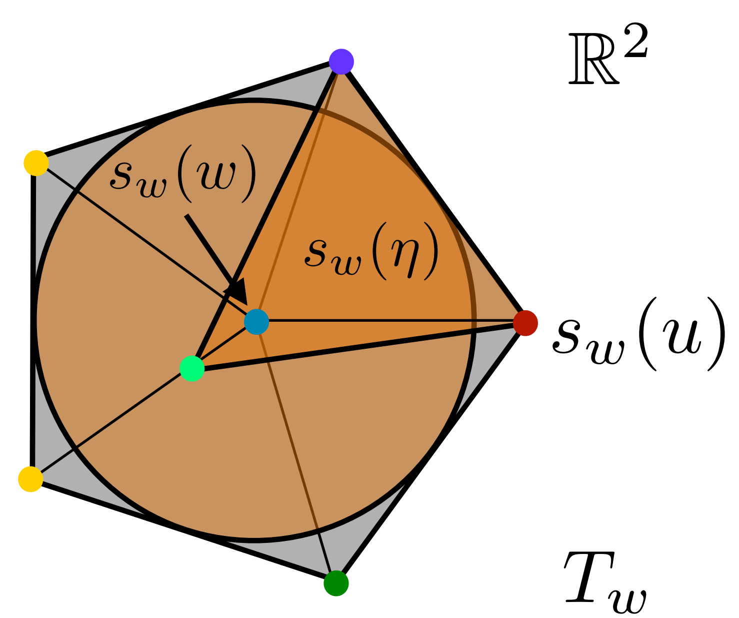

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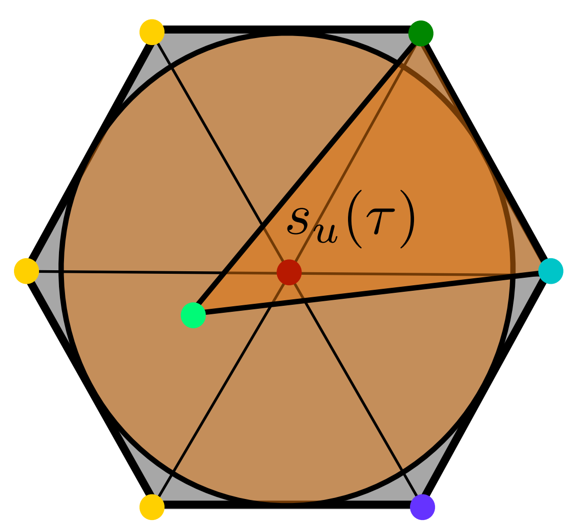

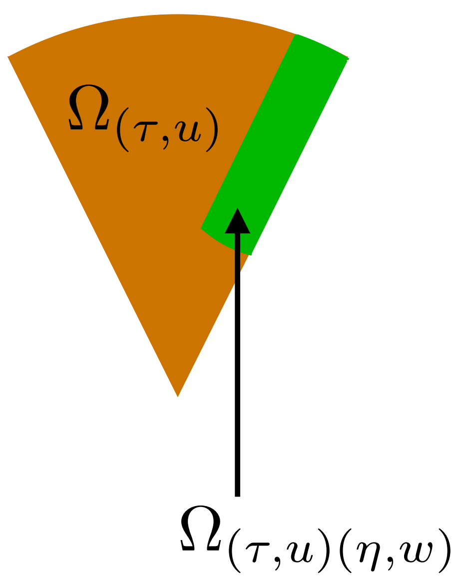

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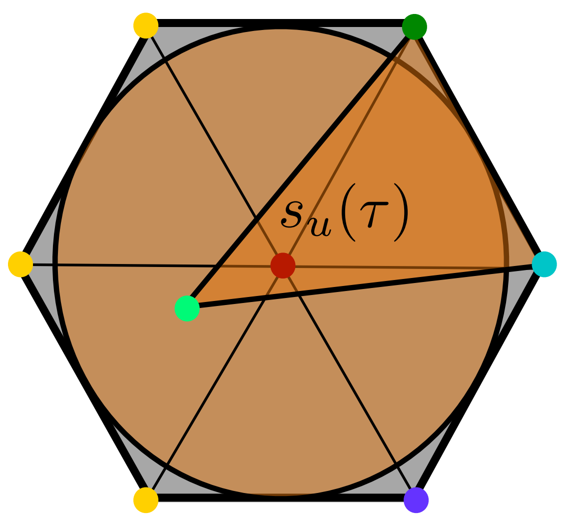

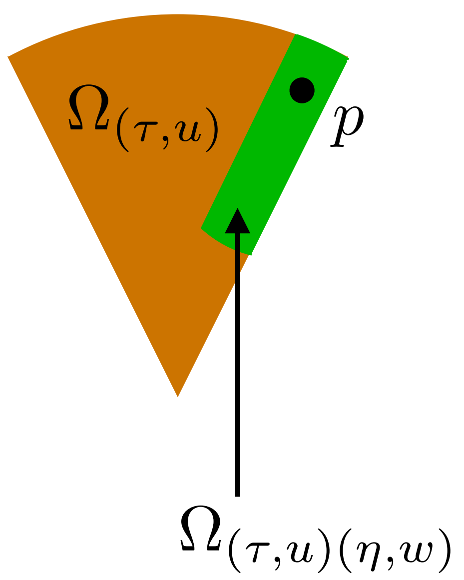

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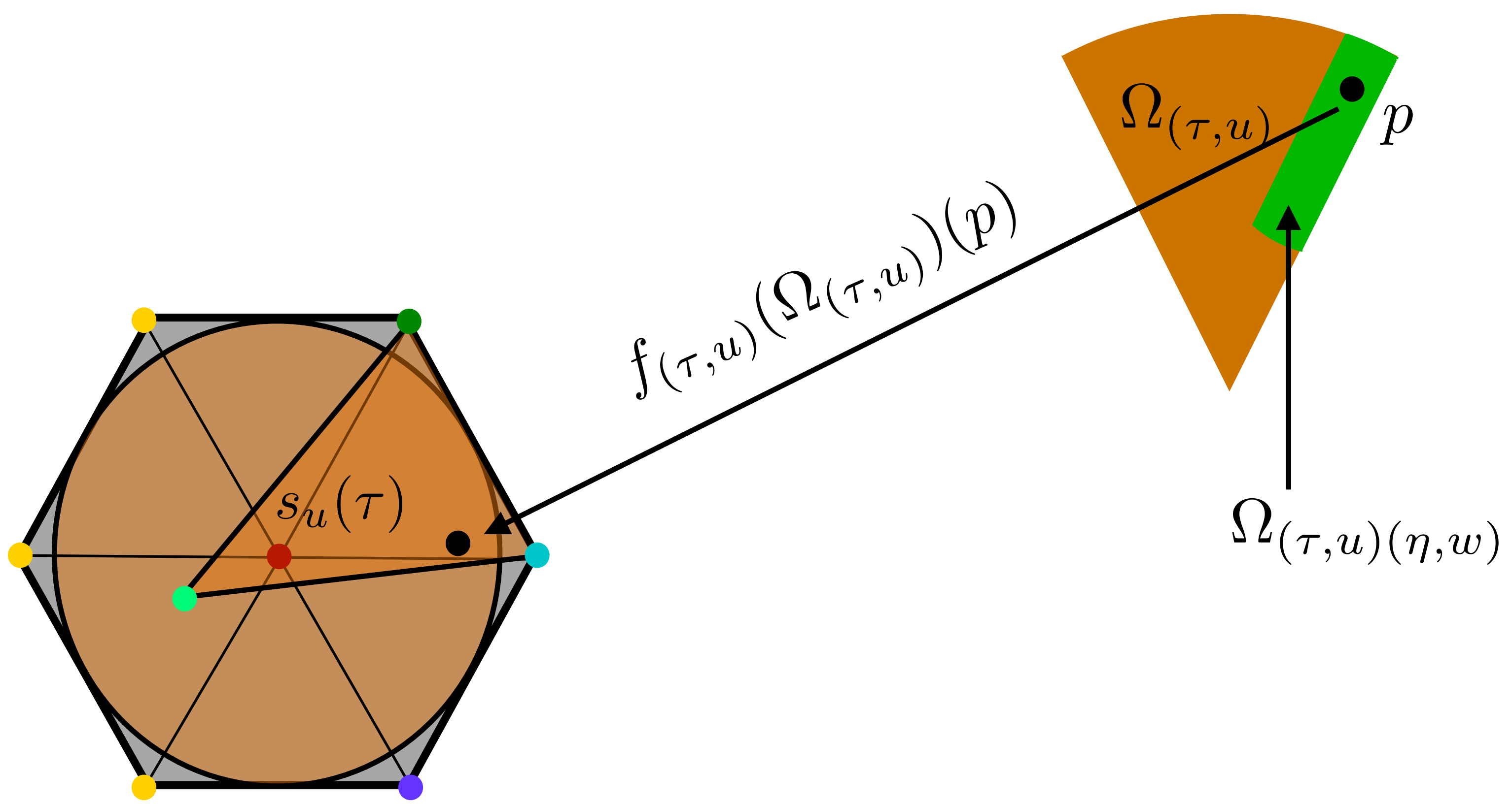

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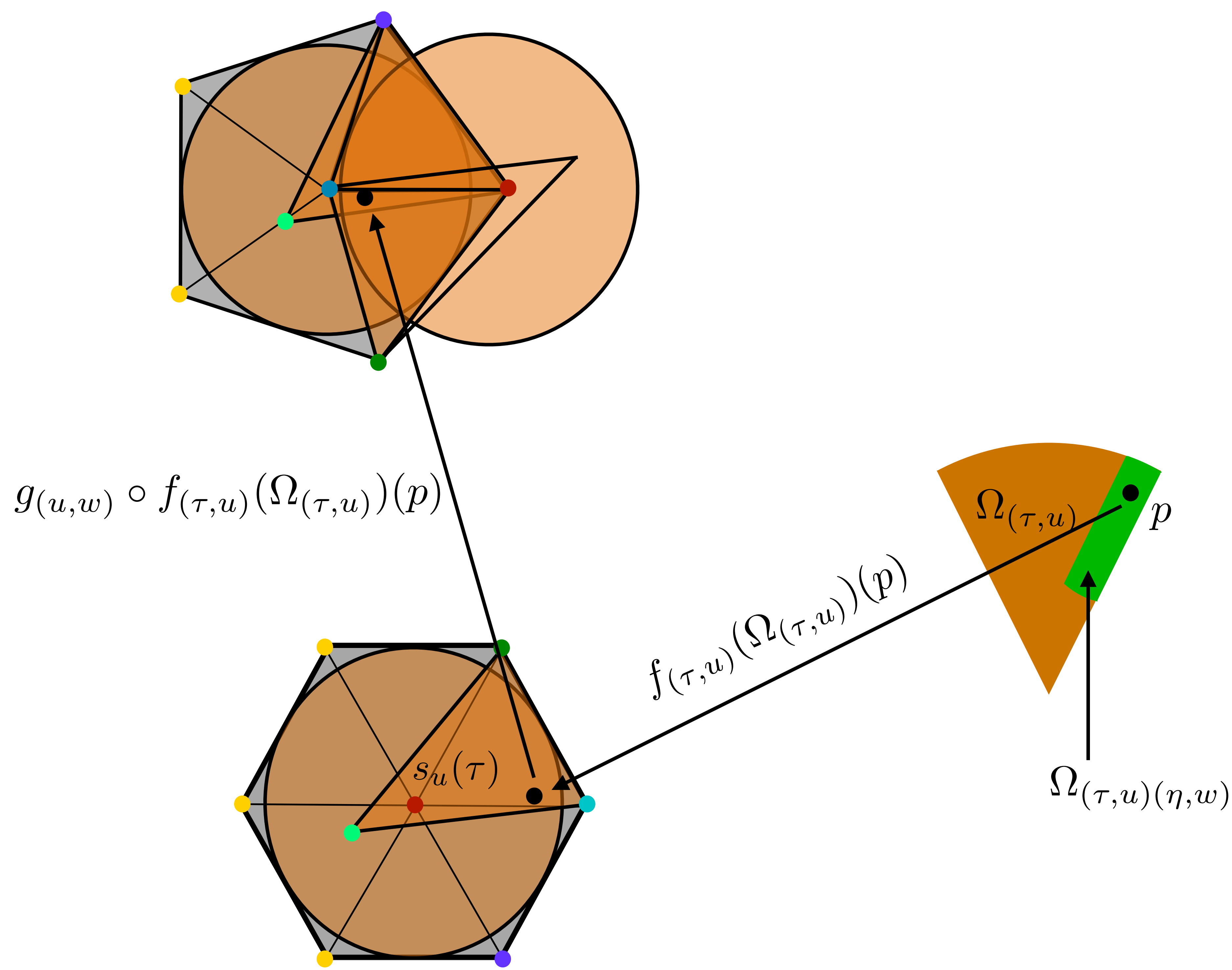

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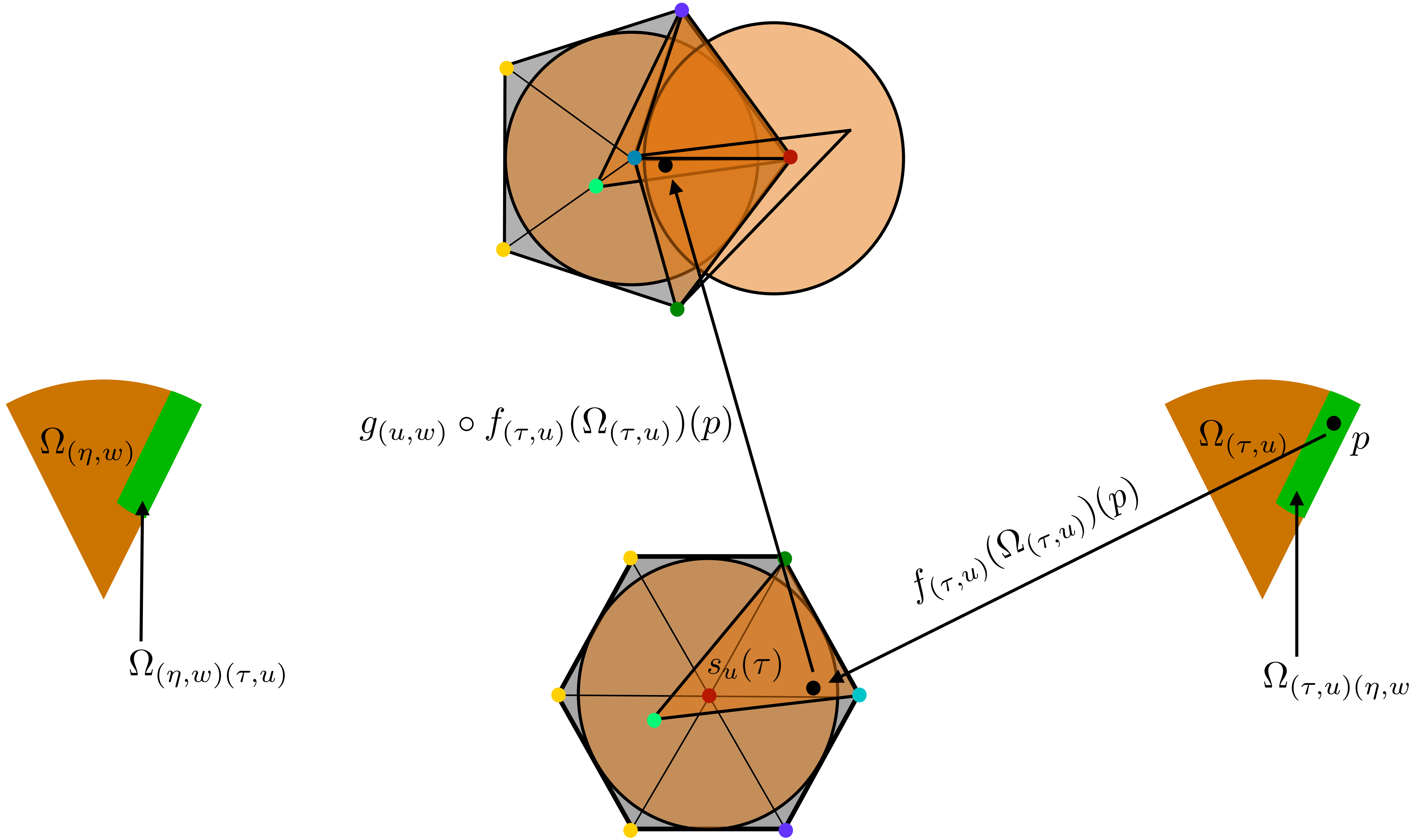

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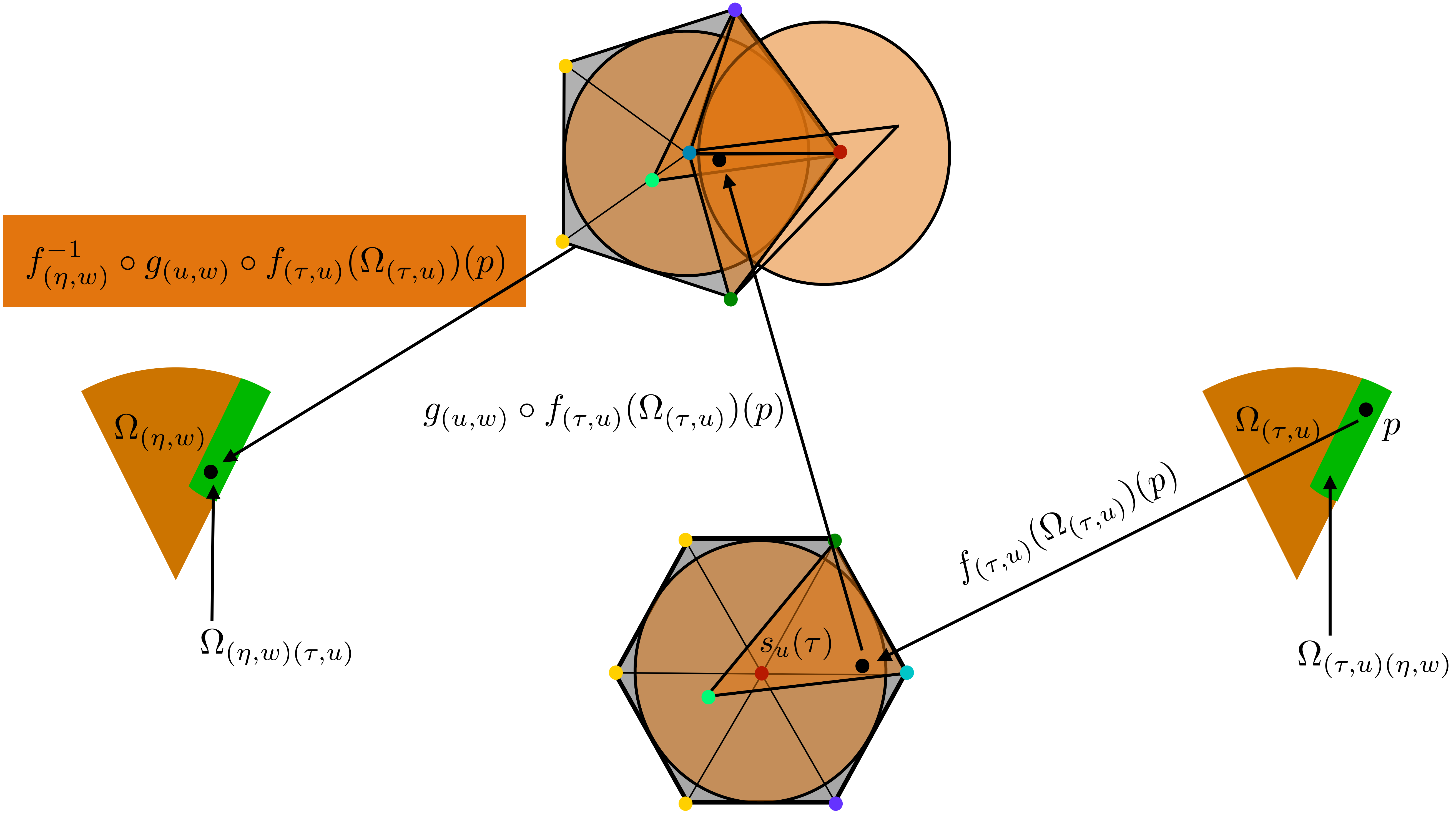

Building a Set of Gluing Data

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For every $((\tau, u), (\eta, w)) \in K$, we define

$$\varphi_{(\eta,w)(\tau,u)} : \Omega_{(\tau,u),(\eta,w)} \rightarrow \varphi_{(\eta,w)(\tau,u)} \, ,$$

the transition function from $\Omega_{(\tau,u)}$ to $\Omega_{(\eta,w)}$, to be

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$$\varphi_{(\eta,w)(\tau,u)}(p) = \begin{cases} f^{-1}_{(\eta,w)} \circ f_{(\tau,u)}(p) & \text{if } u = w \\ \\ f^{-1}_{(\eta,w)} \circ g_{(u,w)} \circ f_{(\tau,u)}(p) & \text{otherwise}\,, \end{cases}$$

for every $p \in \Omega_{(\tau,u)(\eta,w)}$.

Building a Set of Gluing Data

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We can show that the above definition of transition functions satisfies conditions (3)(a)-(c) of the definition of sets of gluing data:

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(a) $\varphi_{ii} = \mathrm{id}_{\Omega_i}$, for all $i \in I$,

(b) $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $(i, j) \in K$, and

(c) for all i, j, and k, if $\Omega_{ji} \cap \Omega_{jk} \neq \emptyset$ then $\varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk}) \subseteq \Omega_{ik}$ and $\varphi_{ki}(x) = \varphi_{kj} \circ \varphi_{ji}(x)$, for all $x \in \varphi_{ji}^{-1}(\Omega_{ji} \cap \Omega_{jk})$.

User's Perspective

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Let t be a triangle in S_T and p be any point in t:

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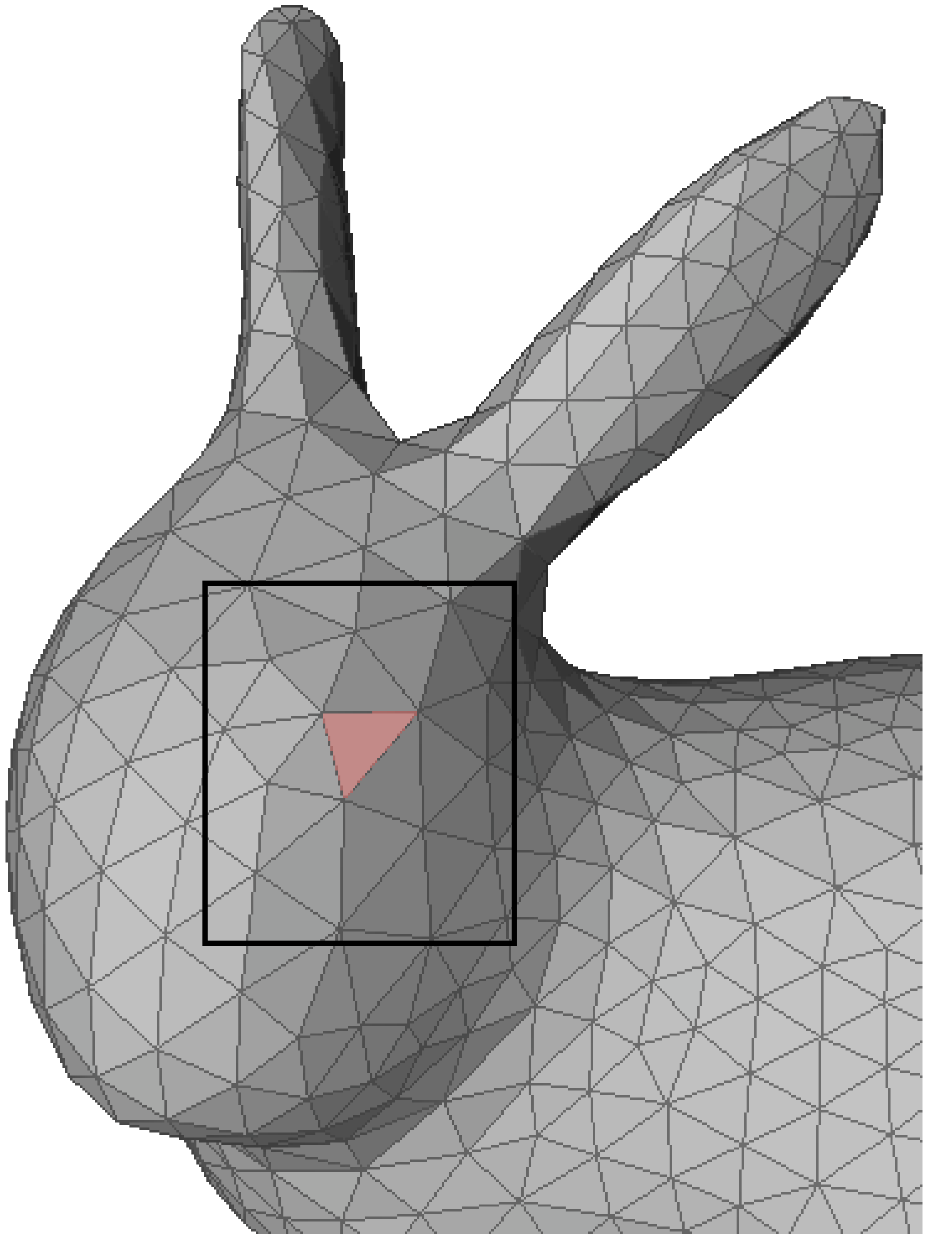

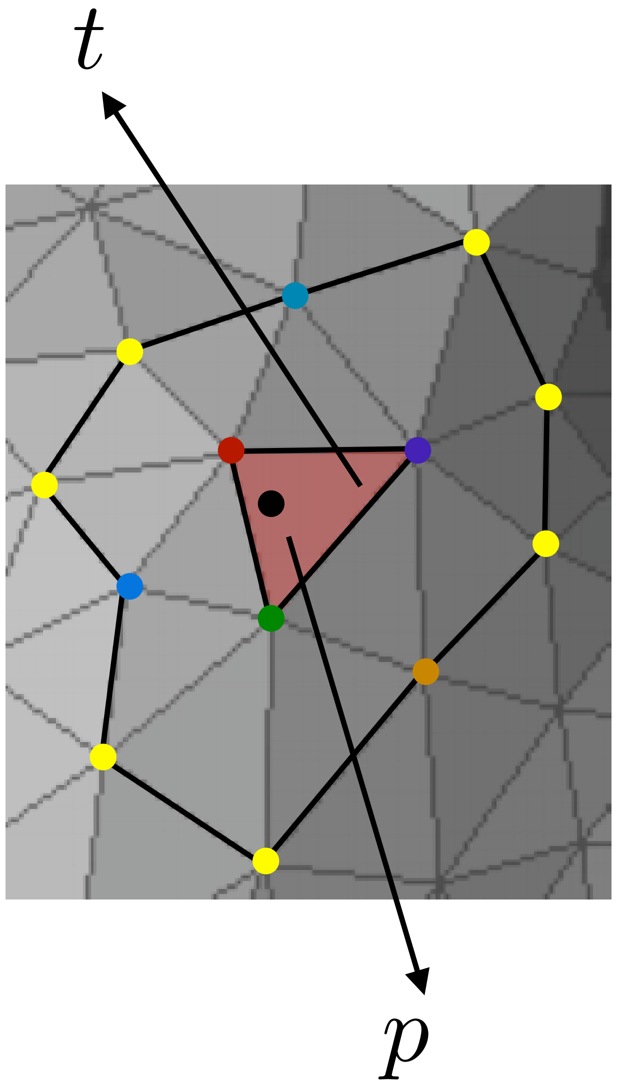

User’s Perspective

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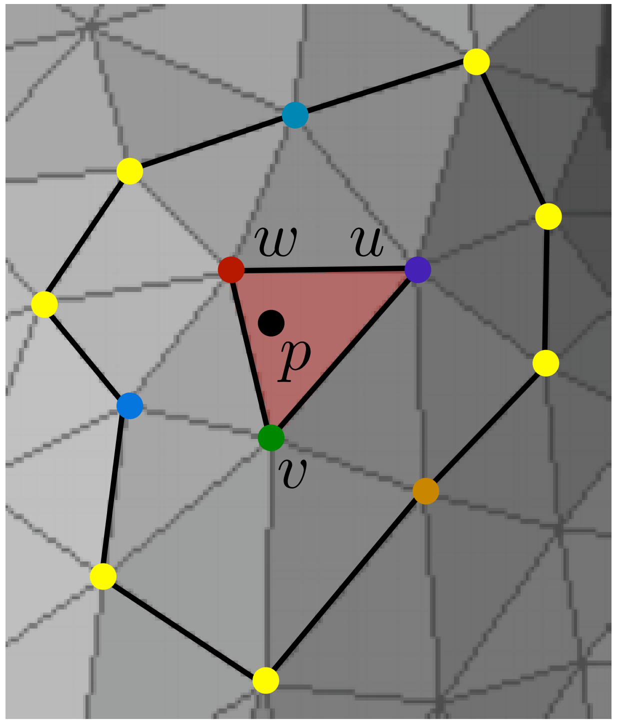
Map p to an equilateral triangle in $\mathbb{R}^2$.



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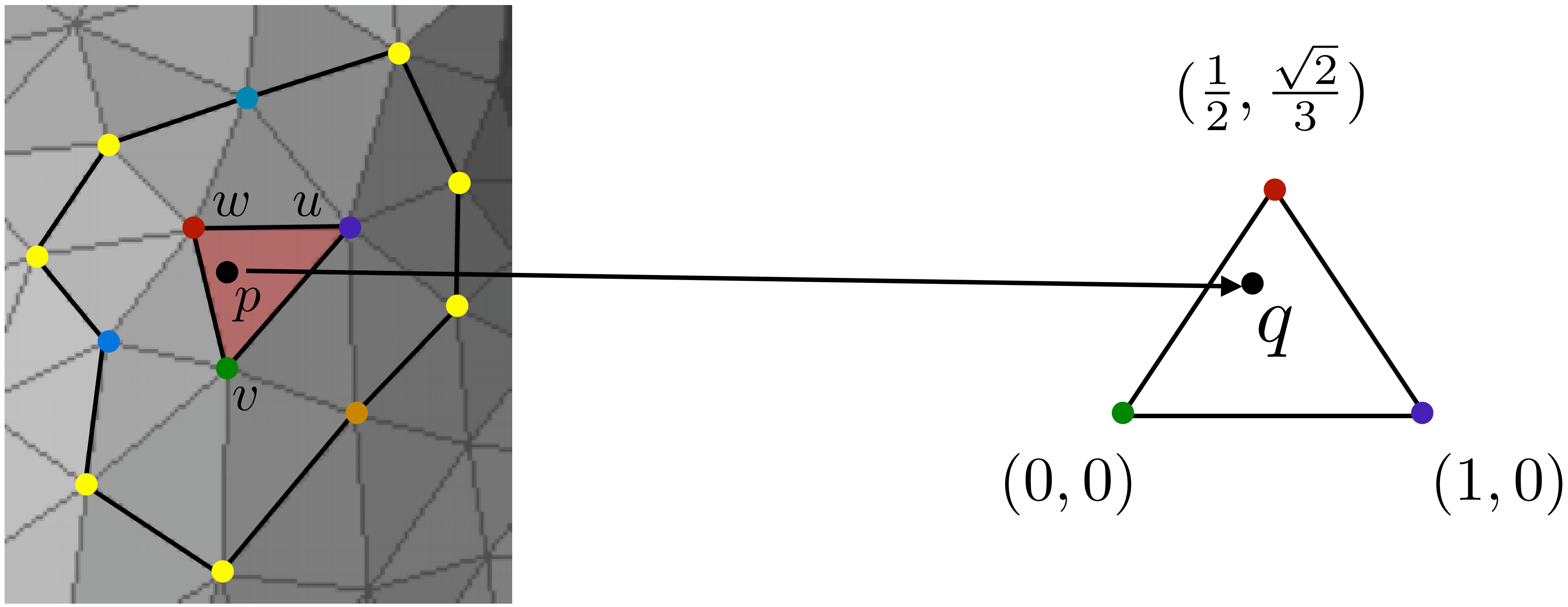

We can do that by using barycentric coordinates.

User’s Perspective

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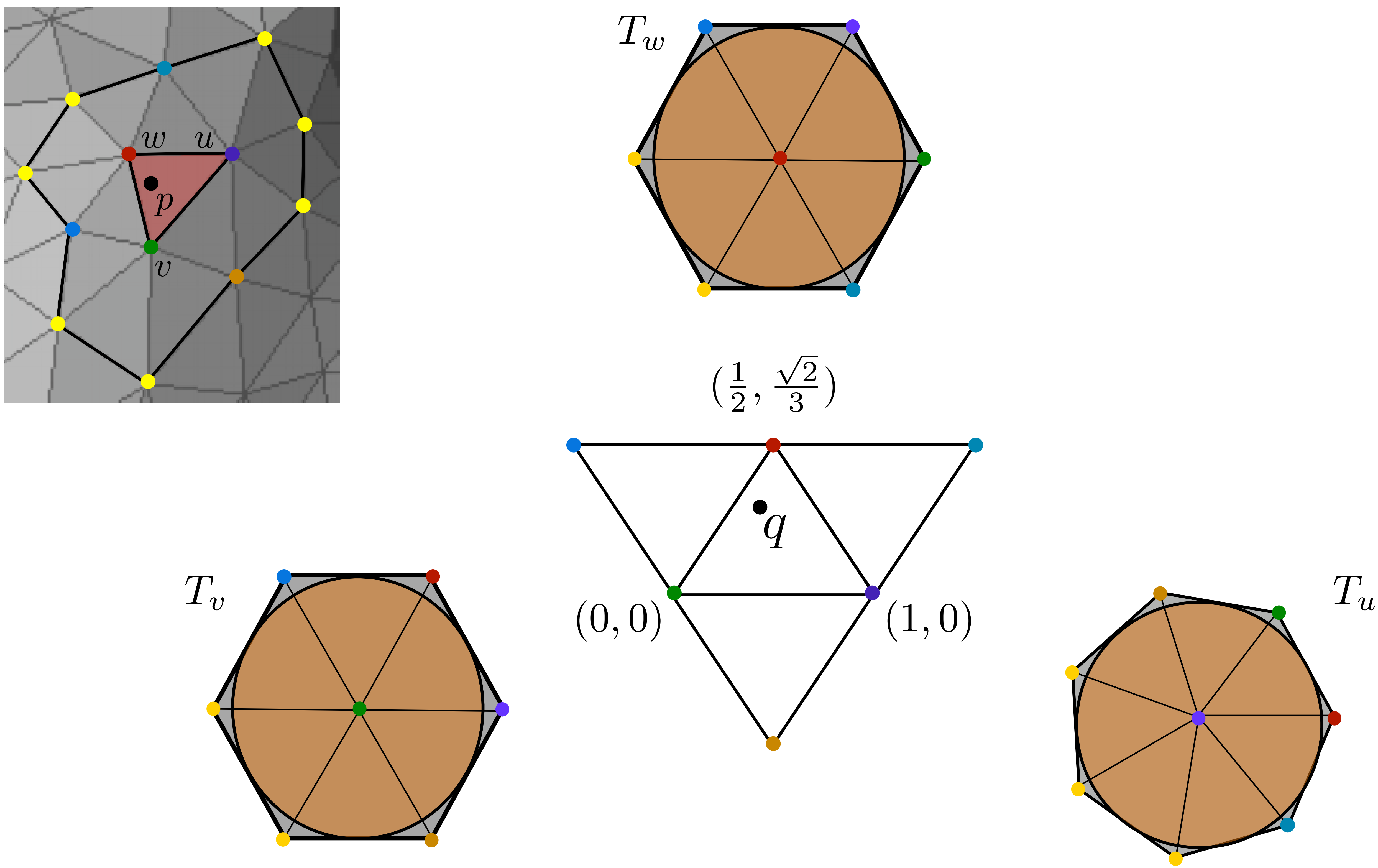

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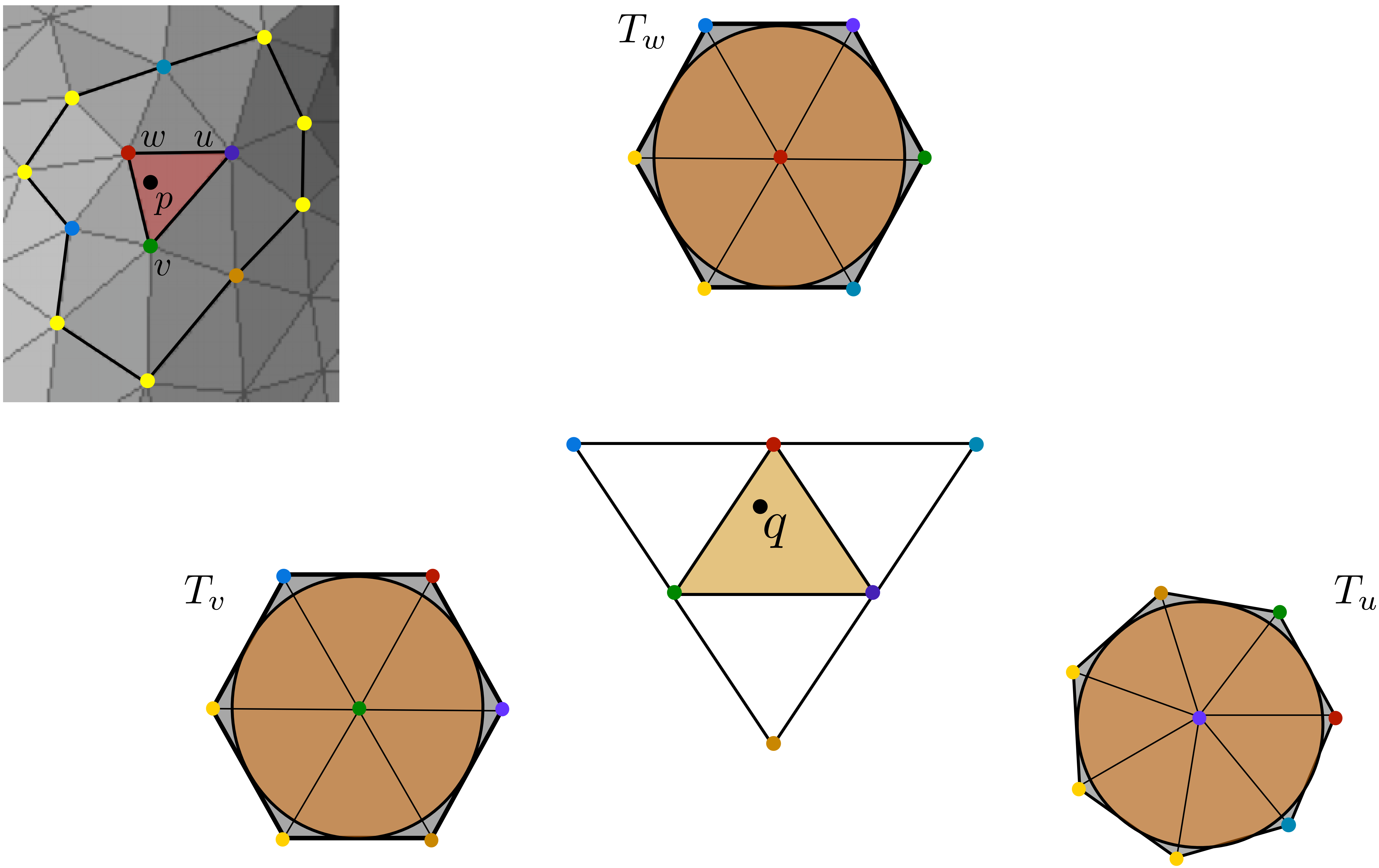

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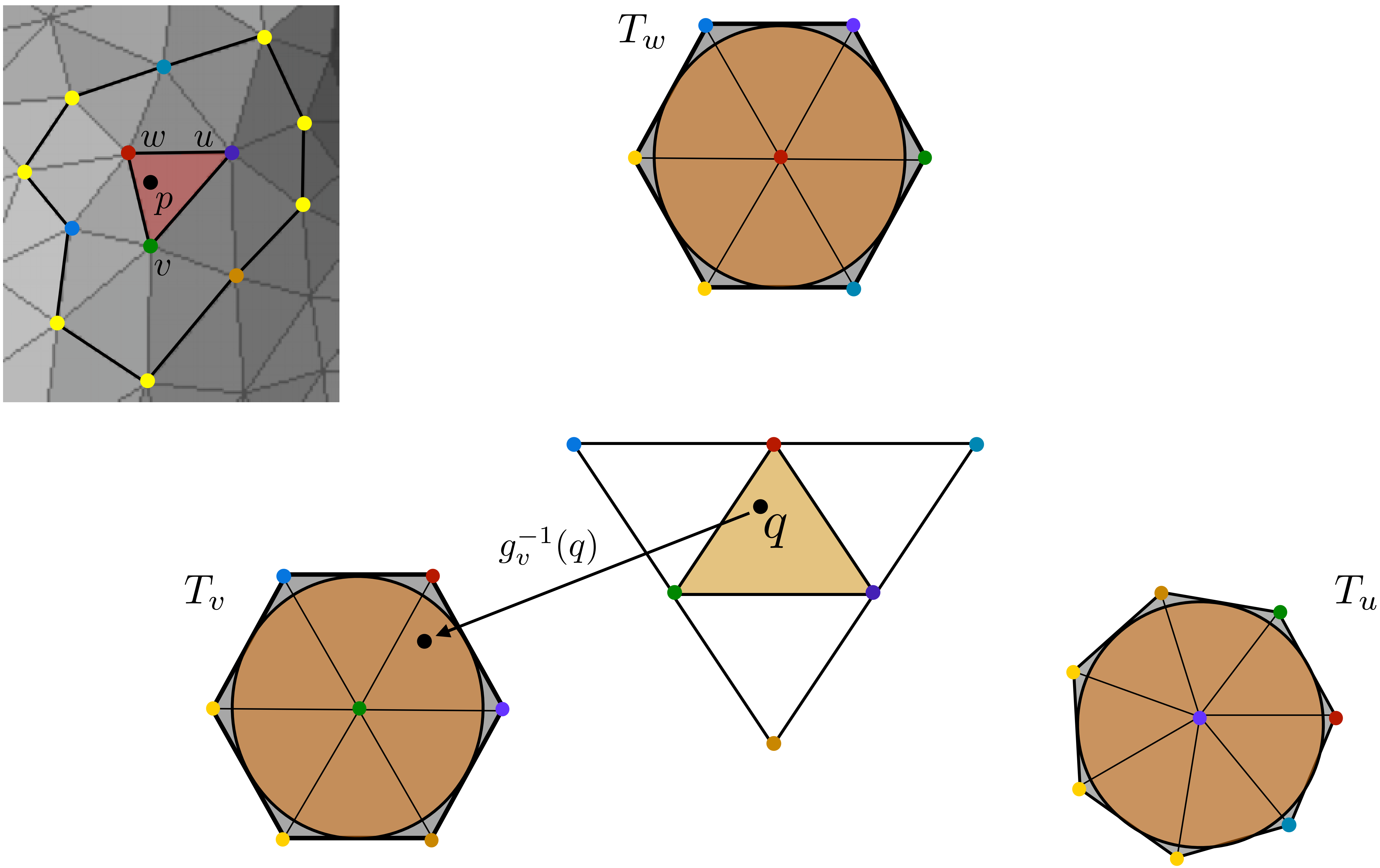

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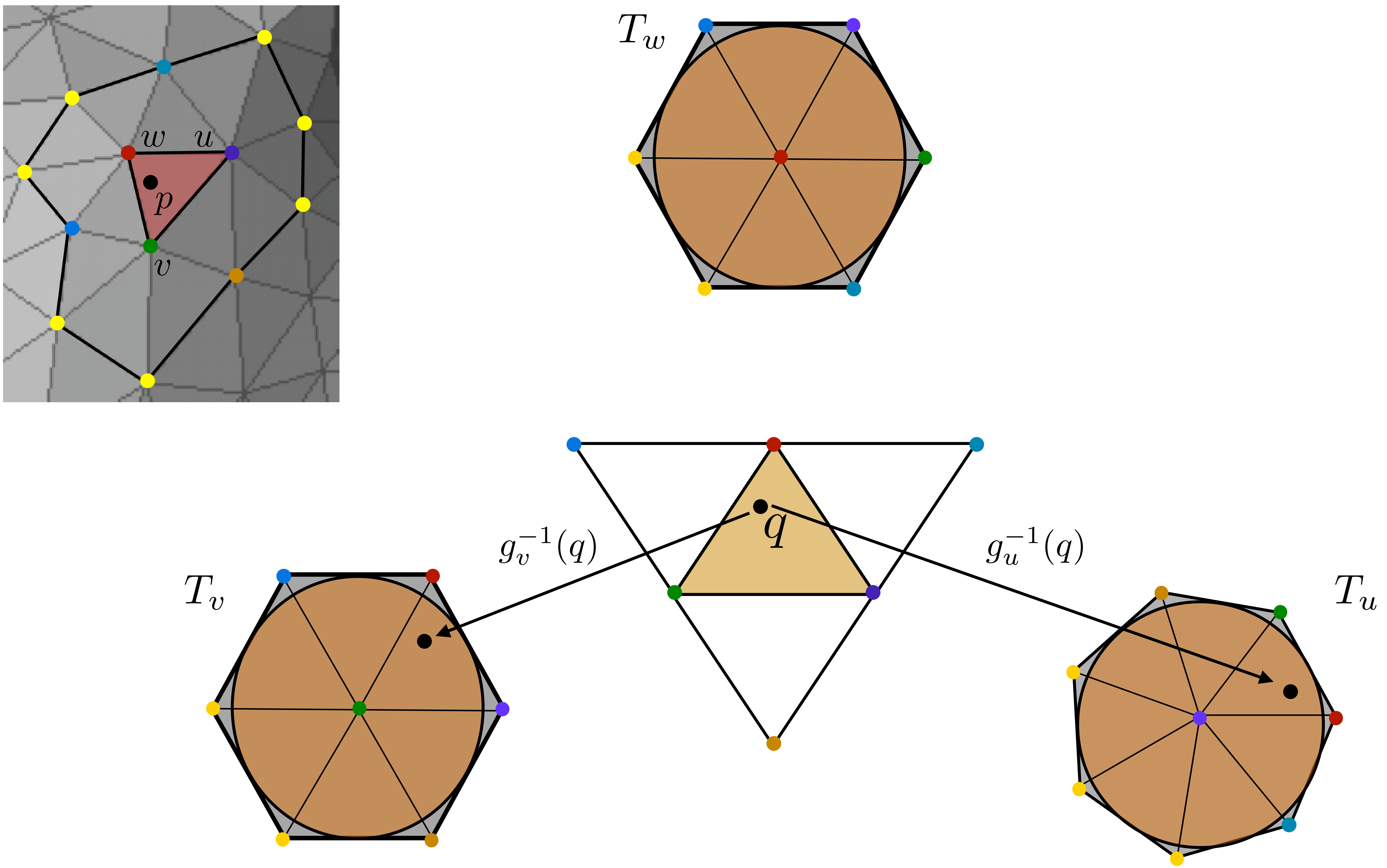

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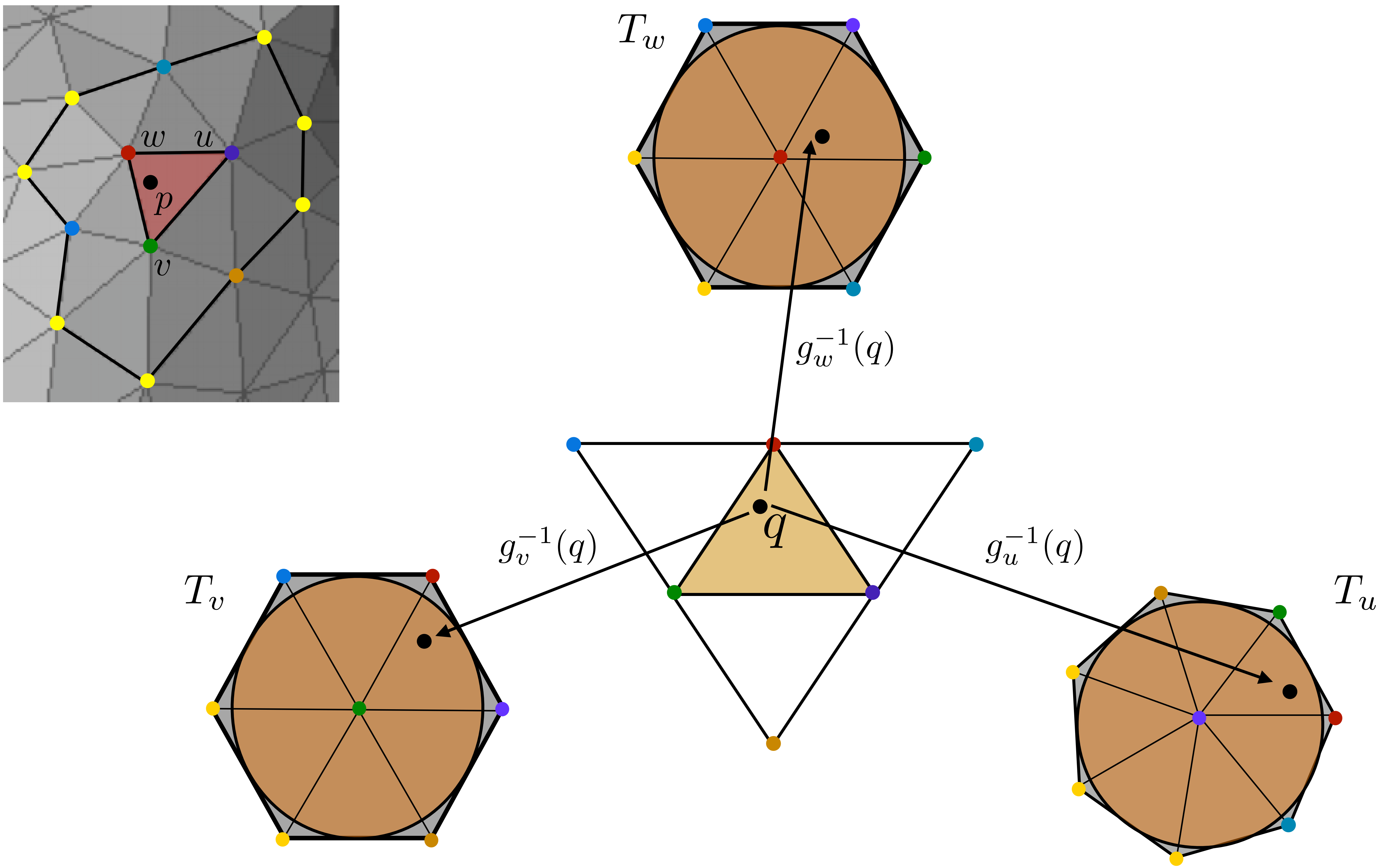

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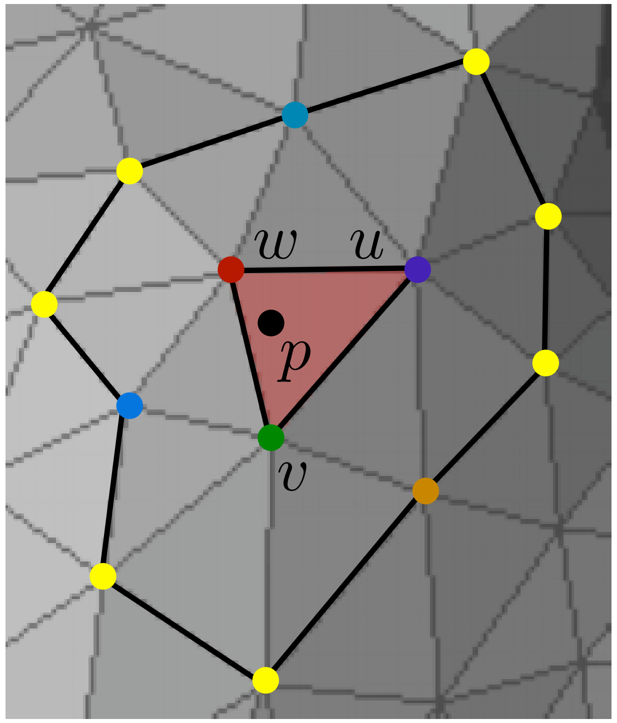

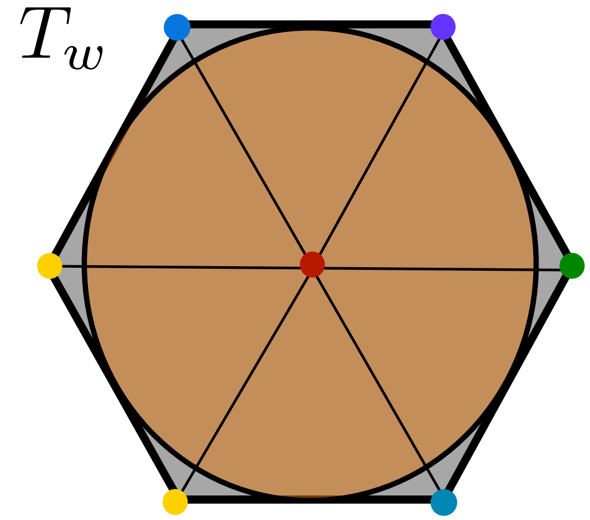

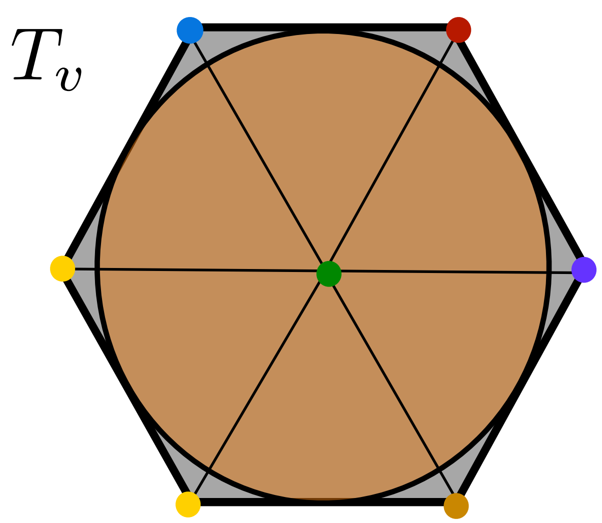

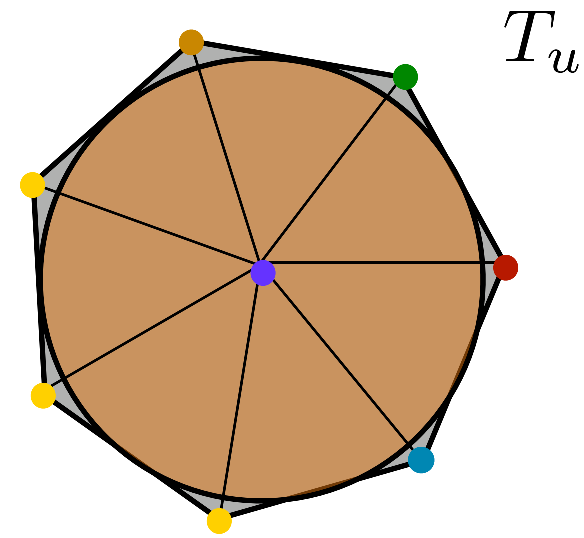

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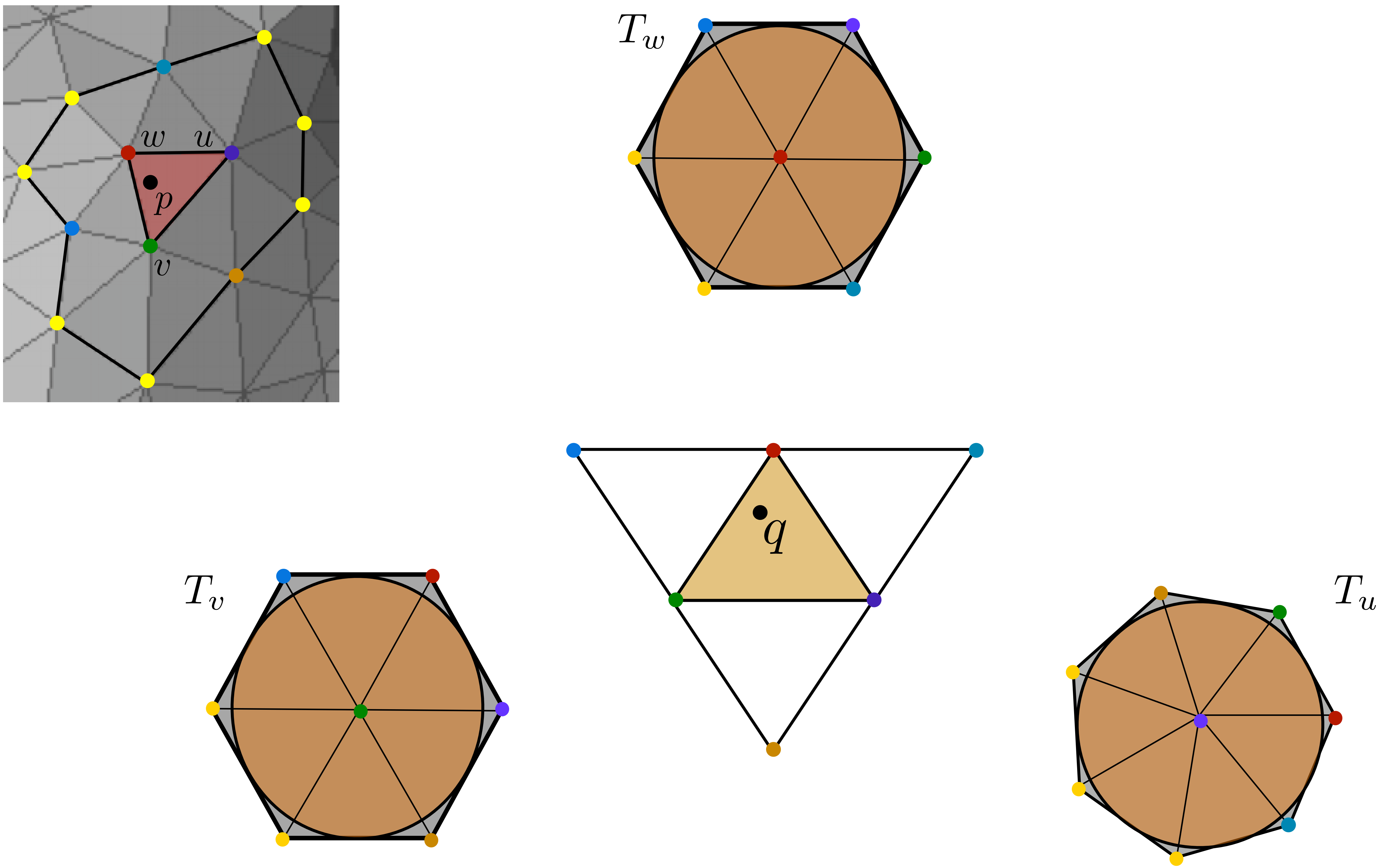

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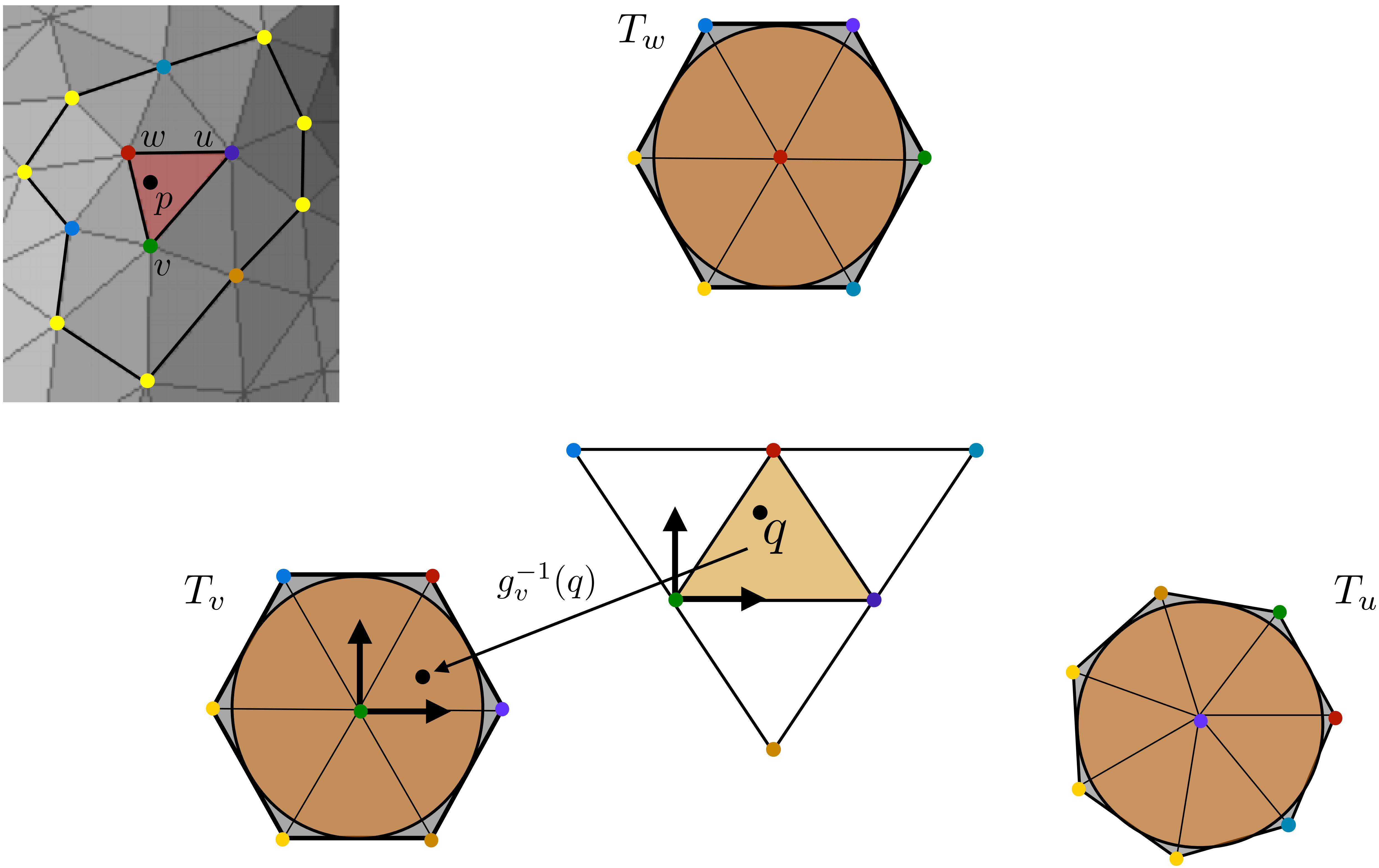

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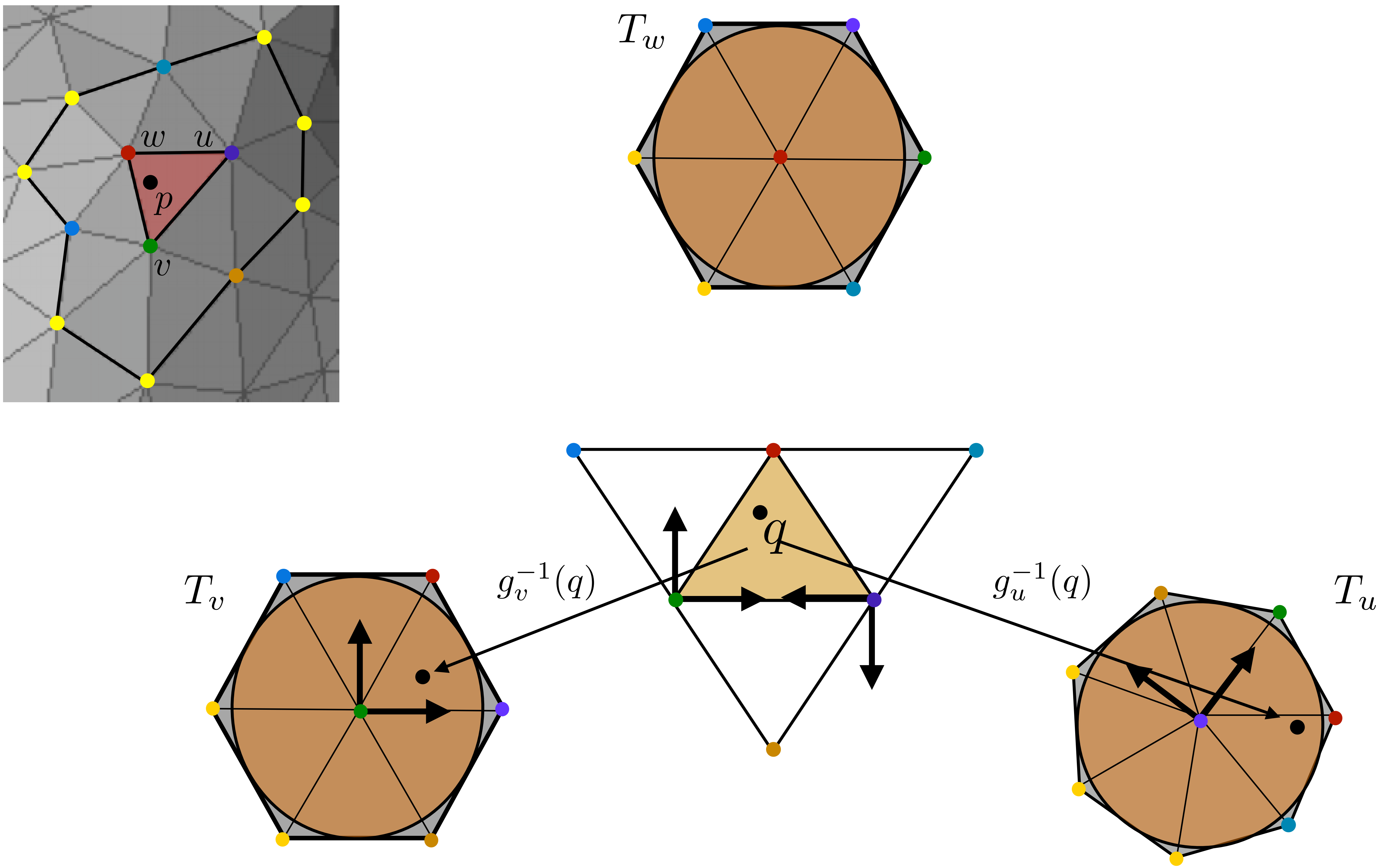

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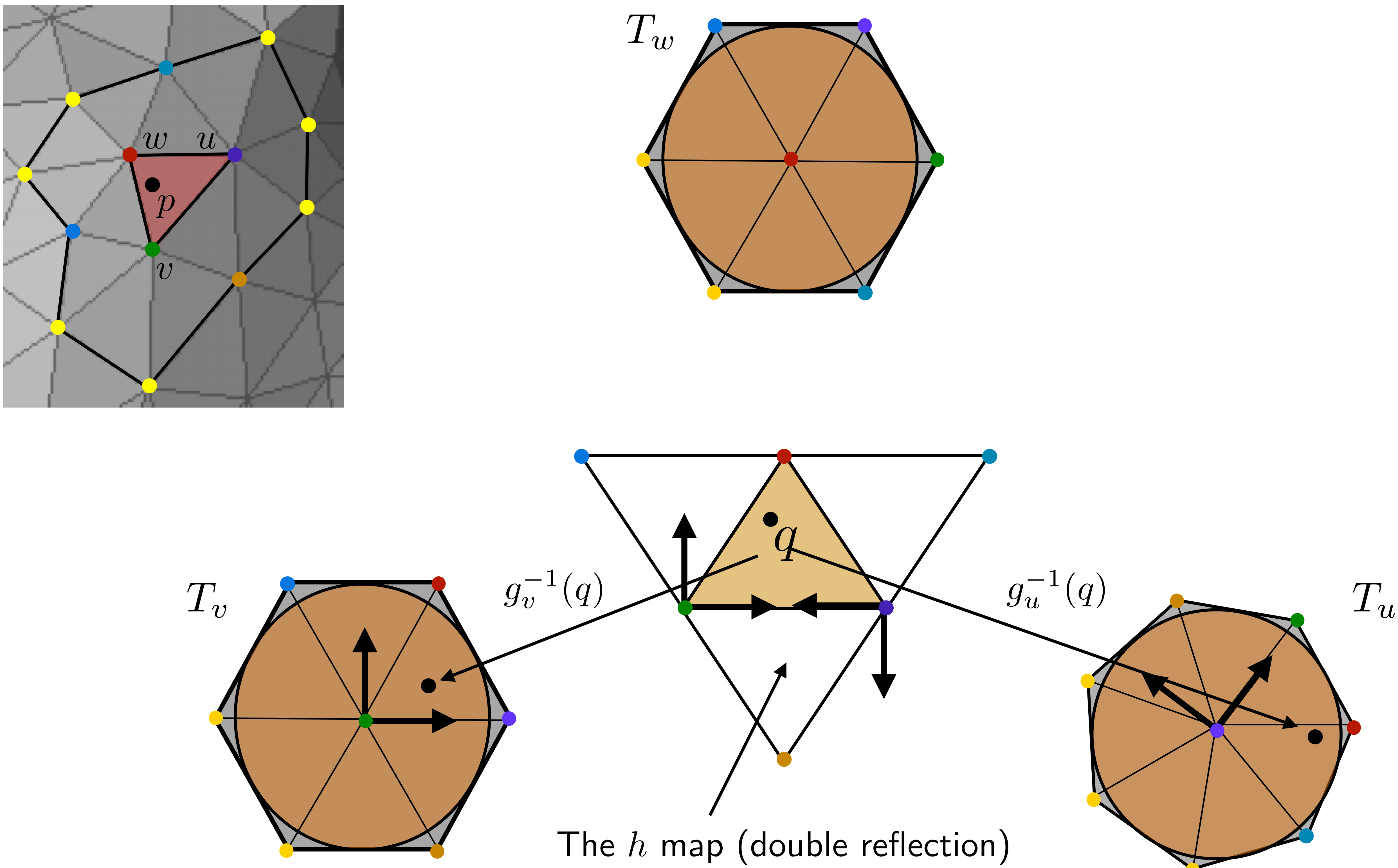

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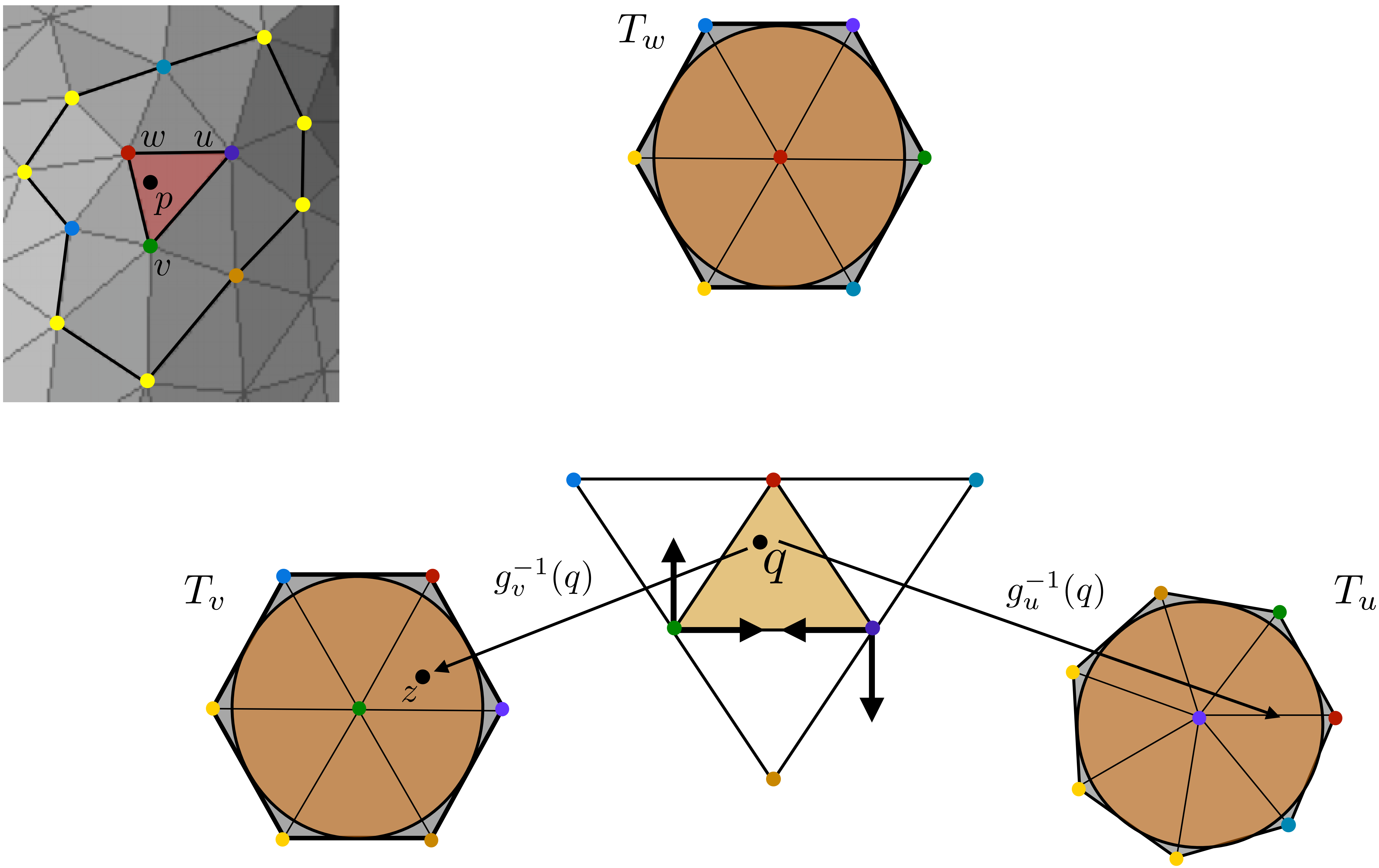

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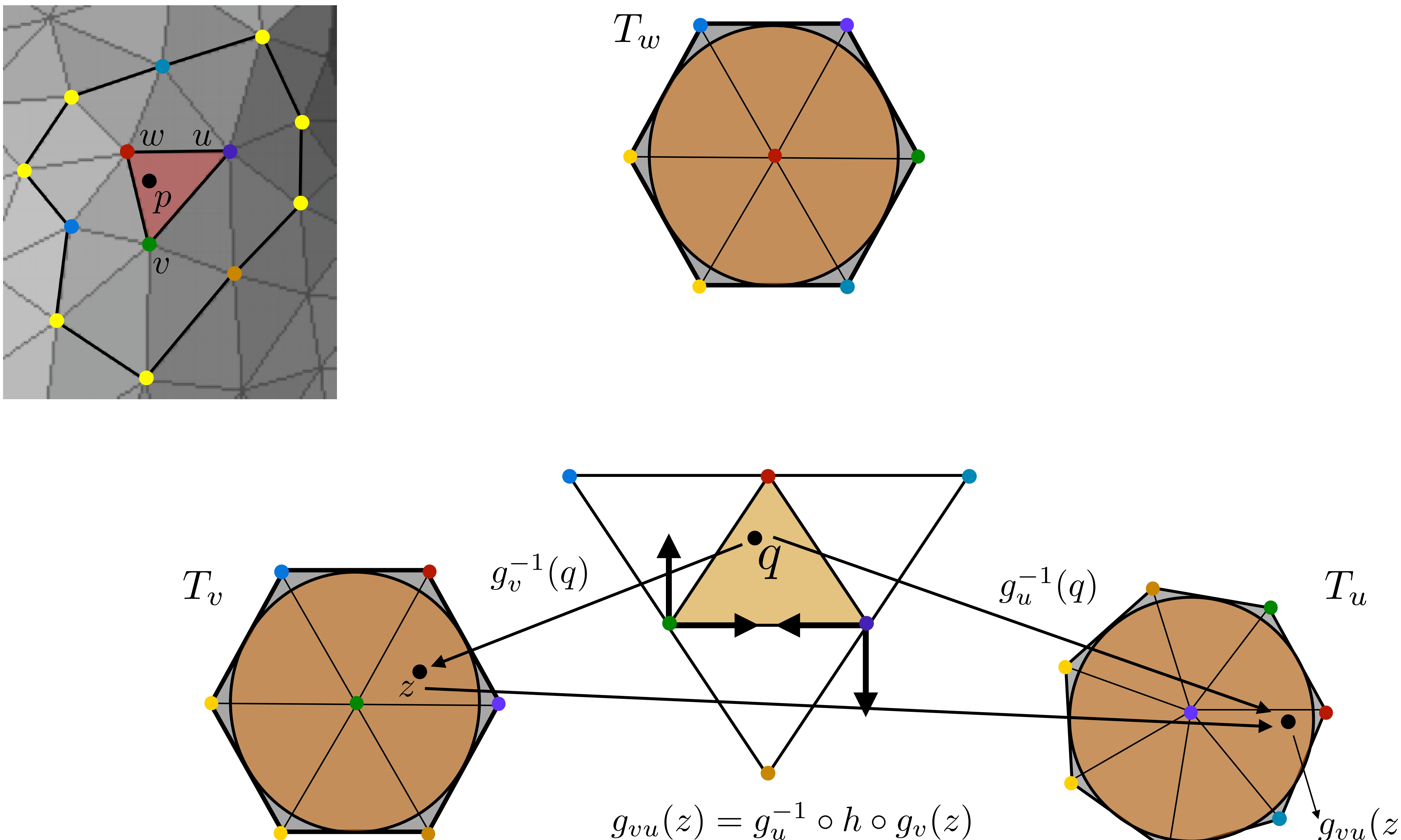

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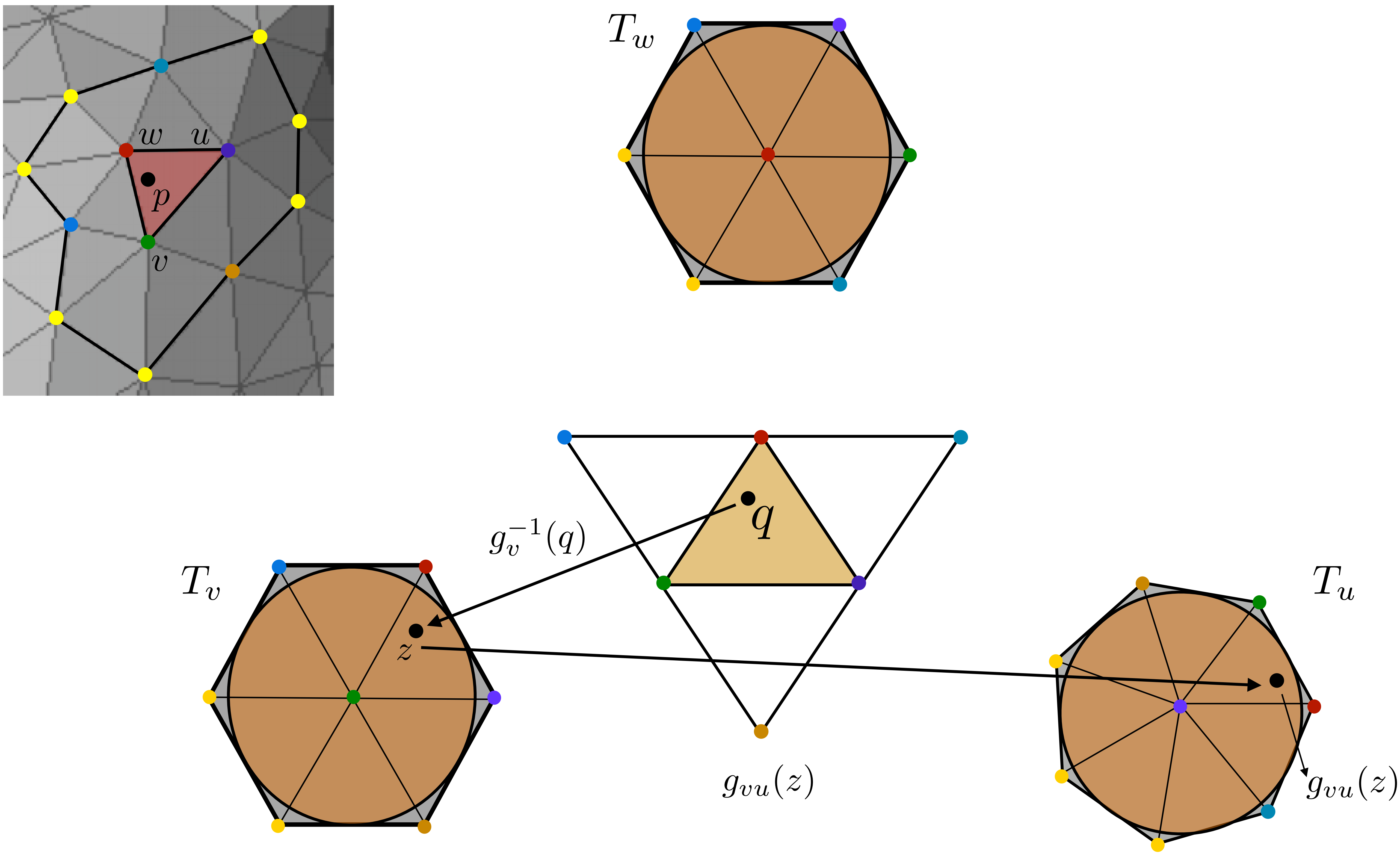

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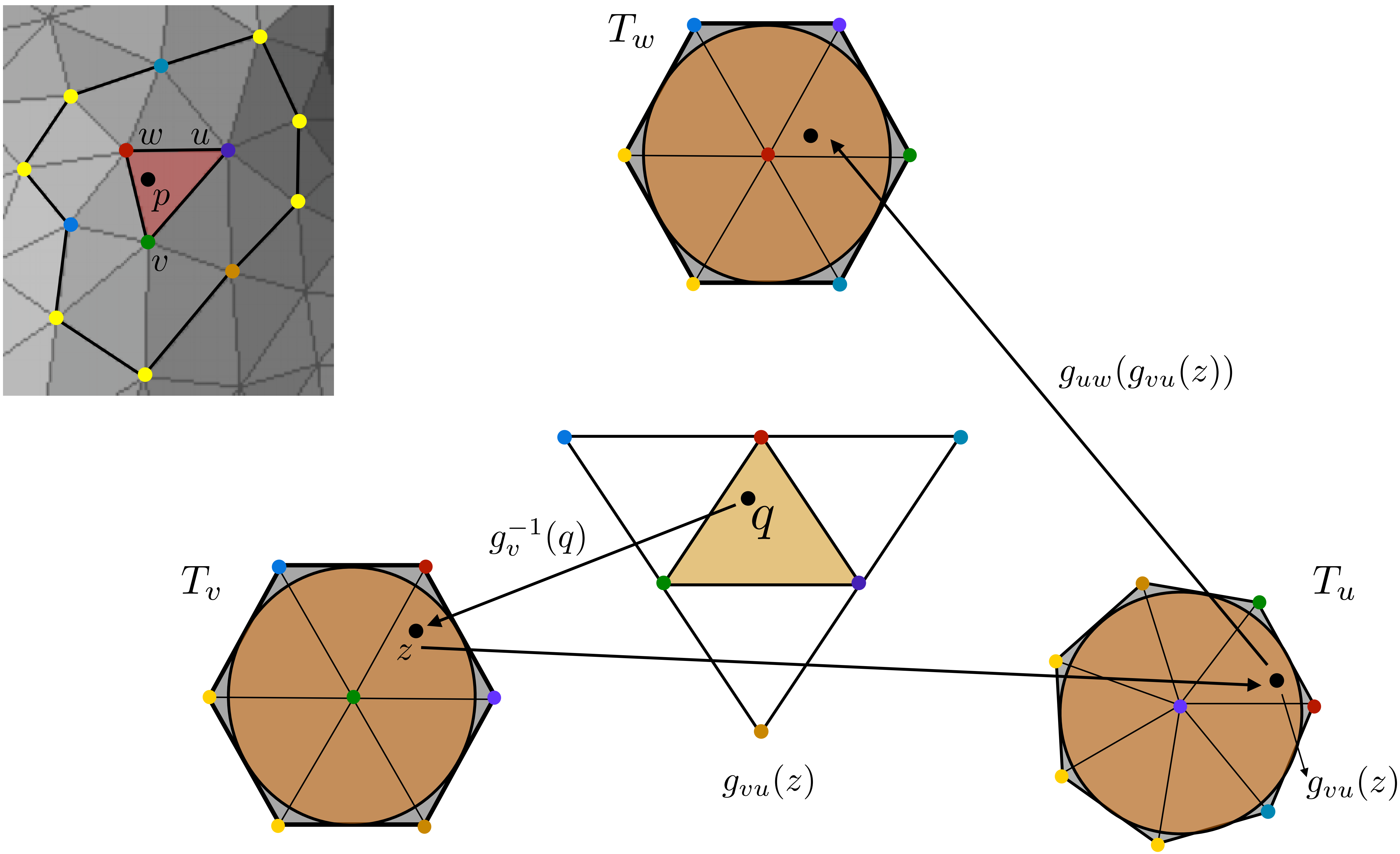

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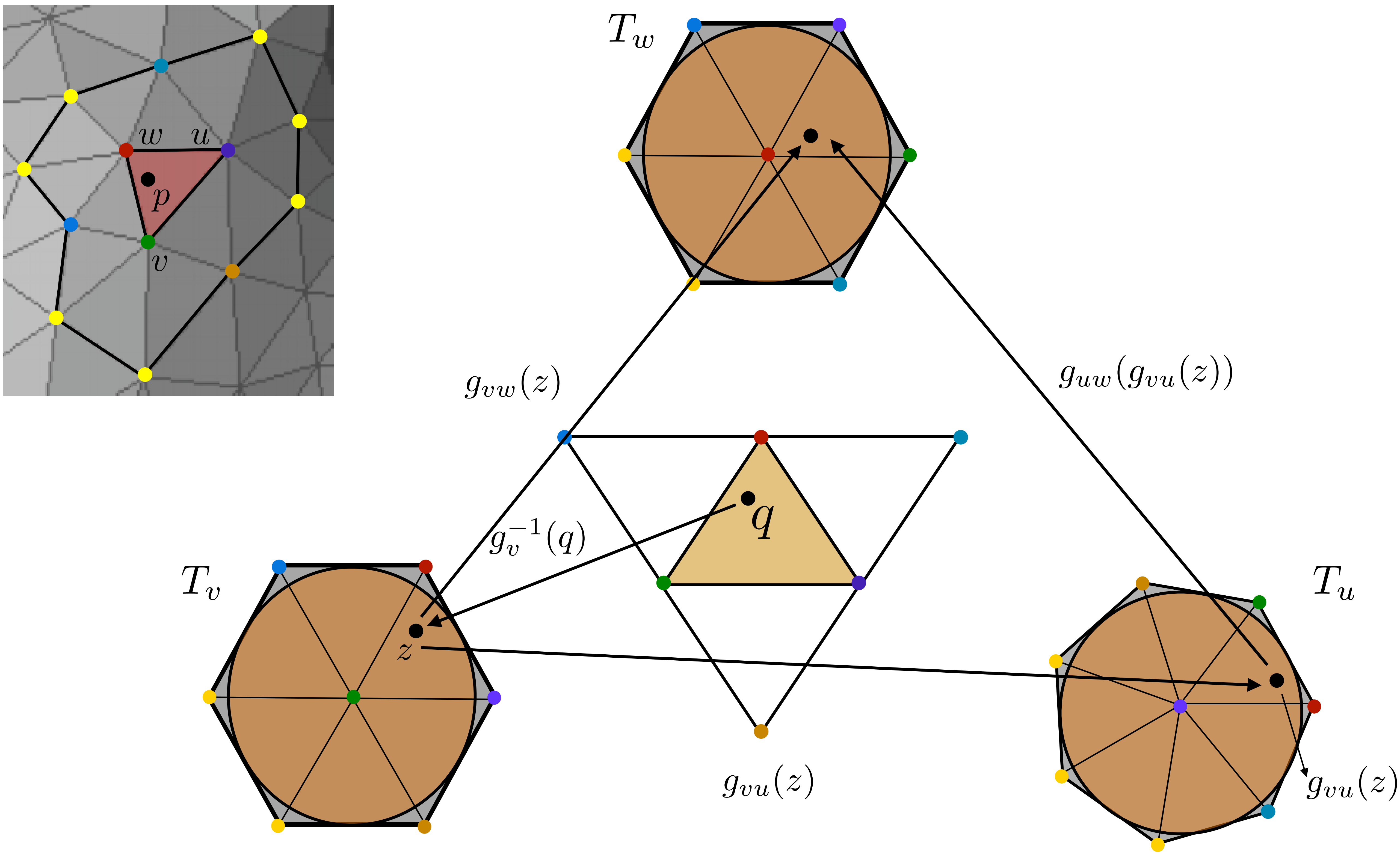

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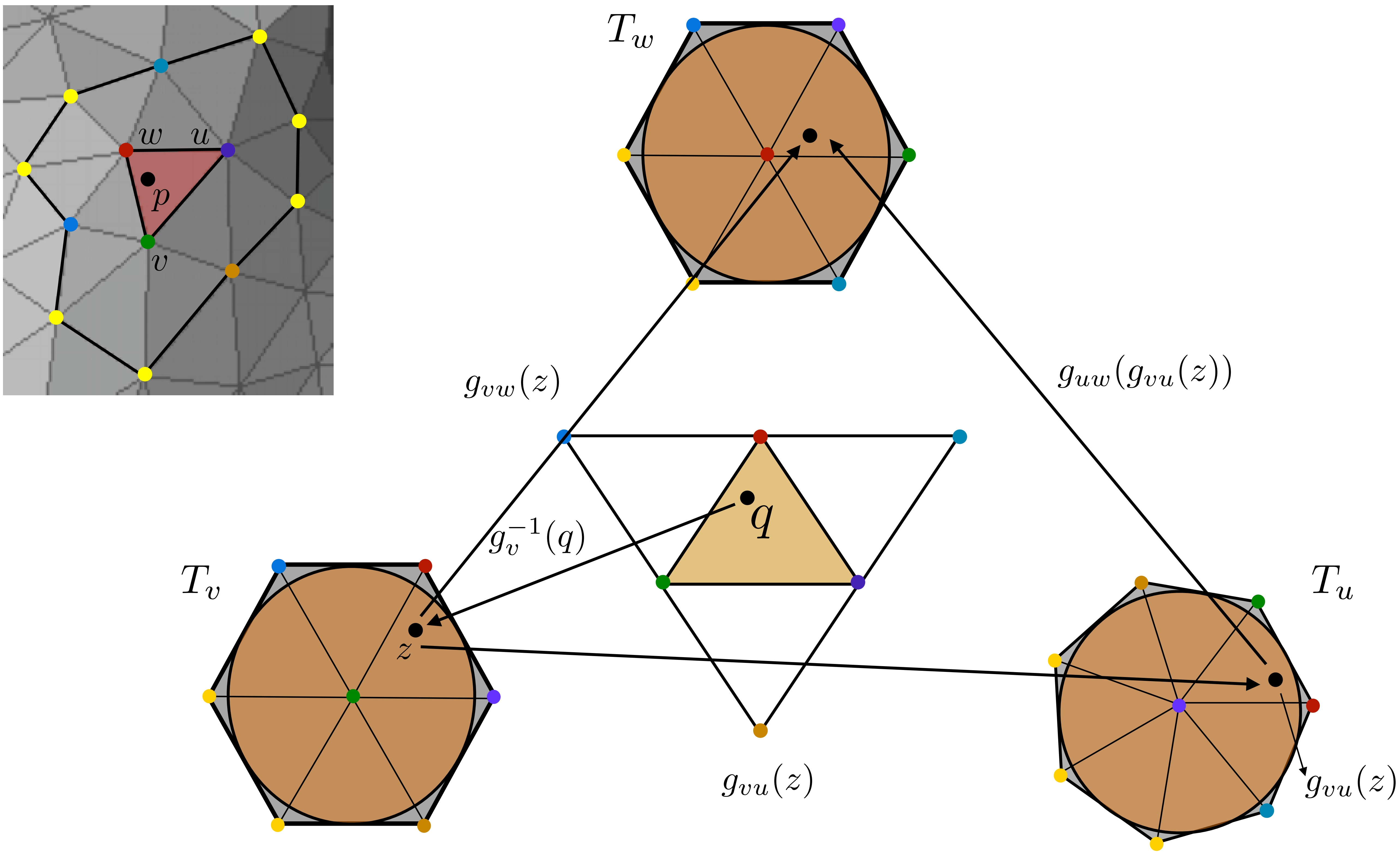

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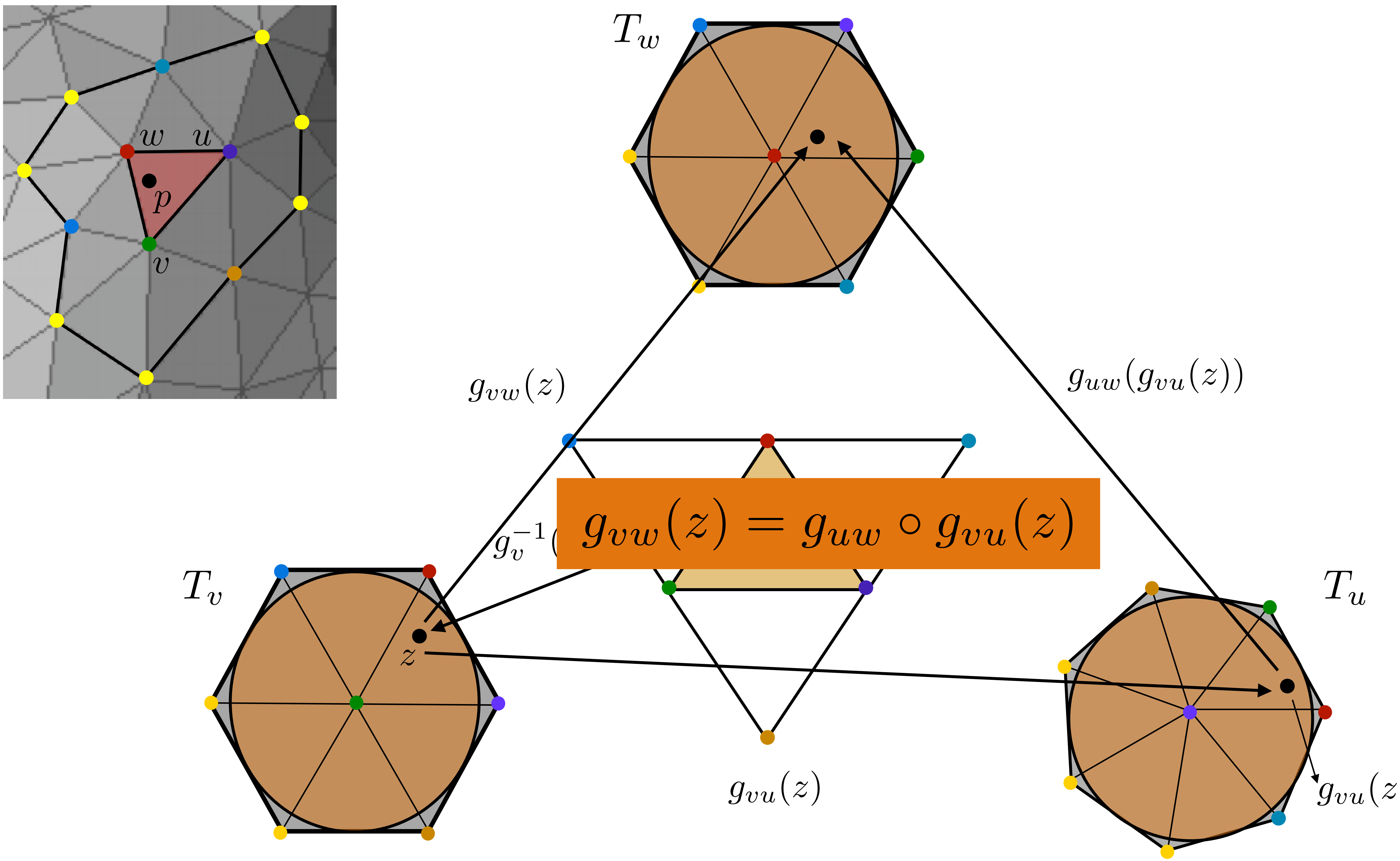

$$g_{vw}(z) = g_{uw} \circ g_{vu}(z)$$

Building Parametrizations

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For each $(\sigma, v) \in I$, we define a **weight function**,

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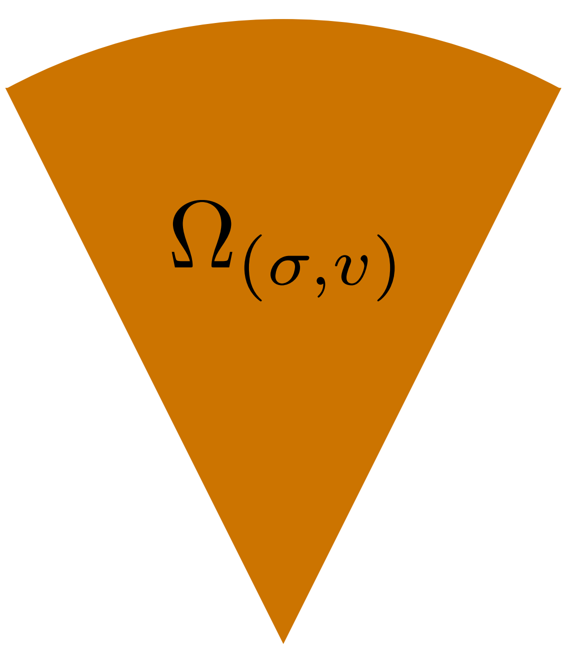

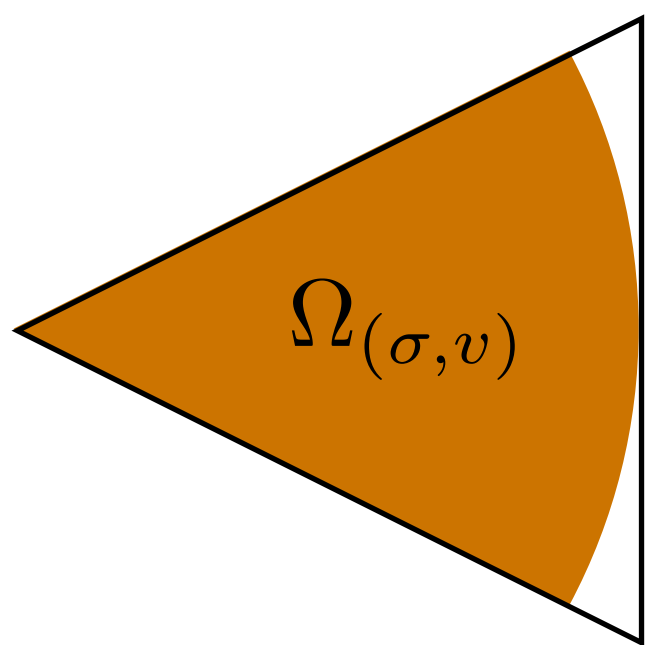

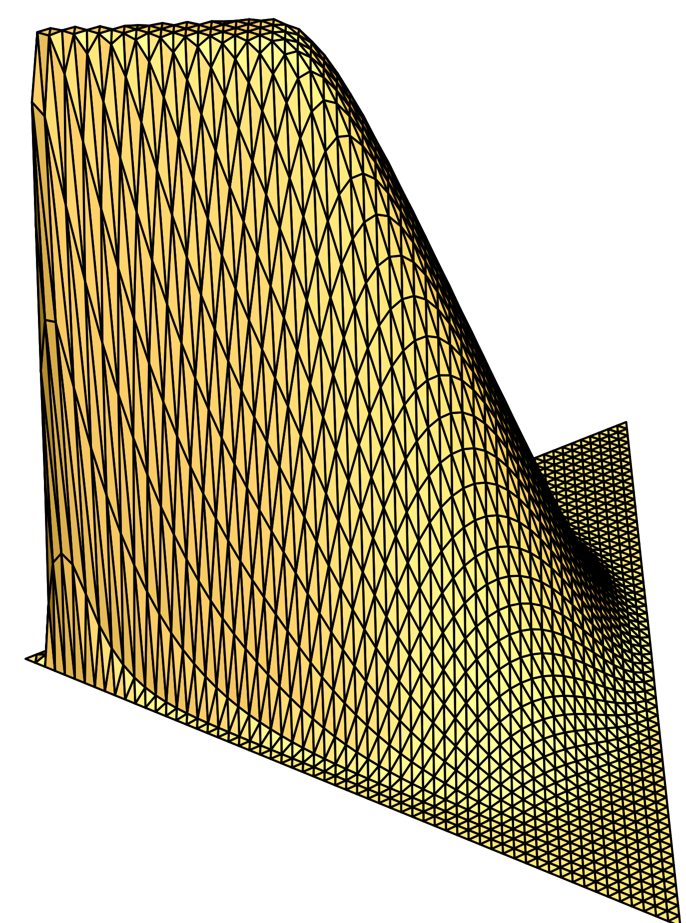

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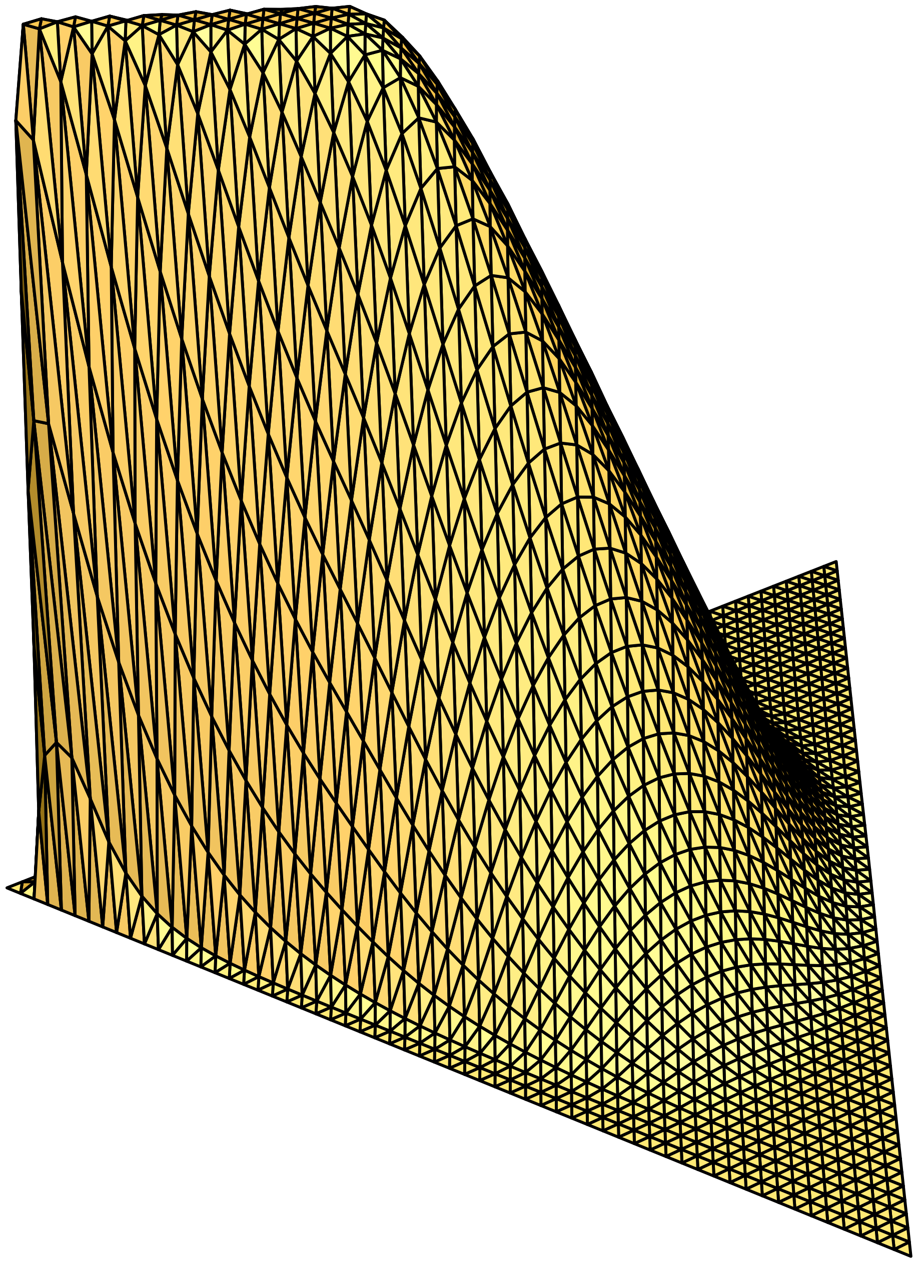

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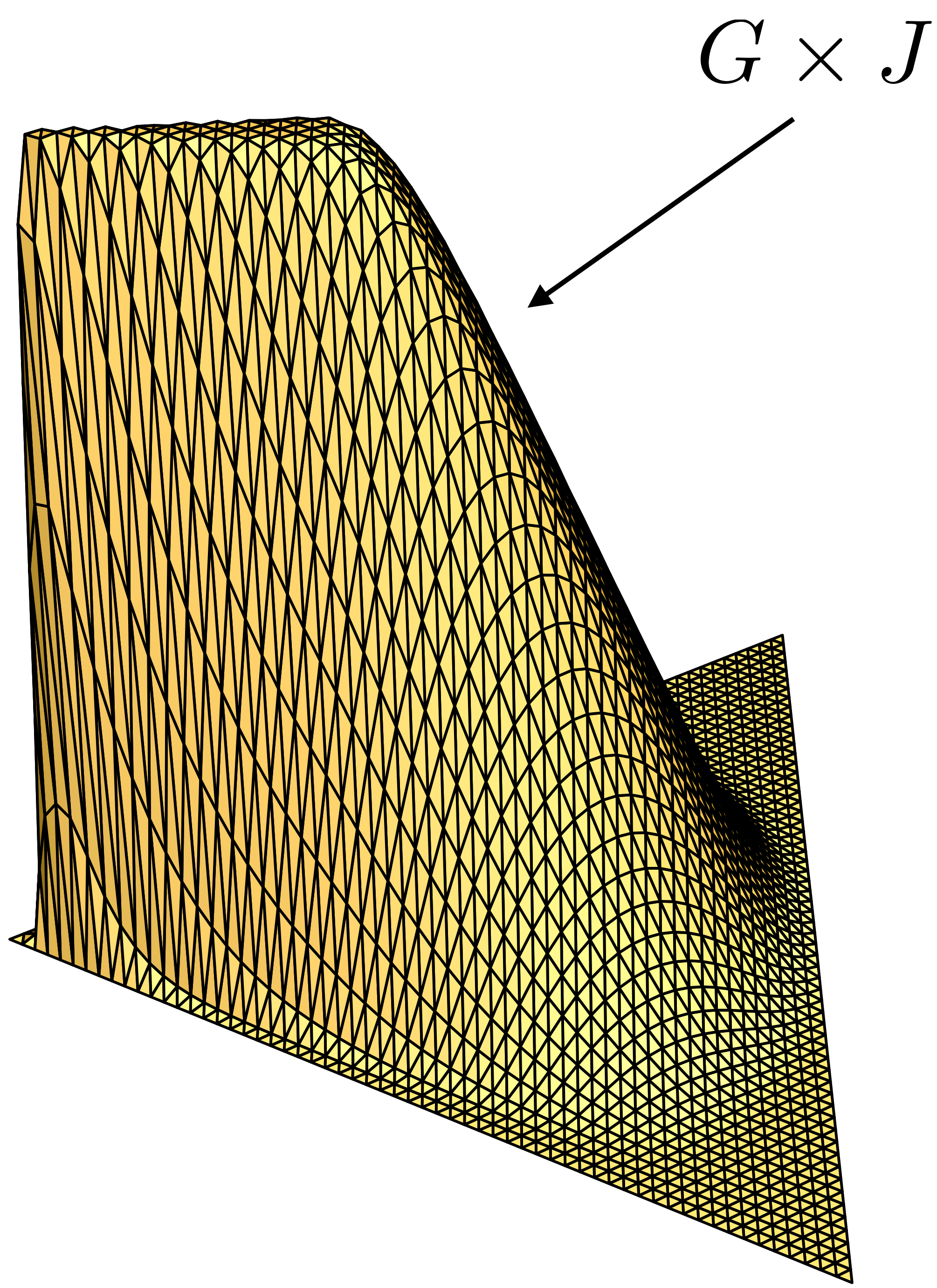

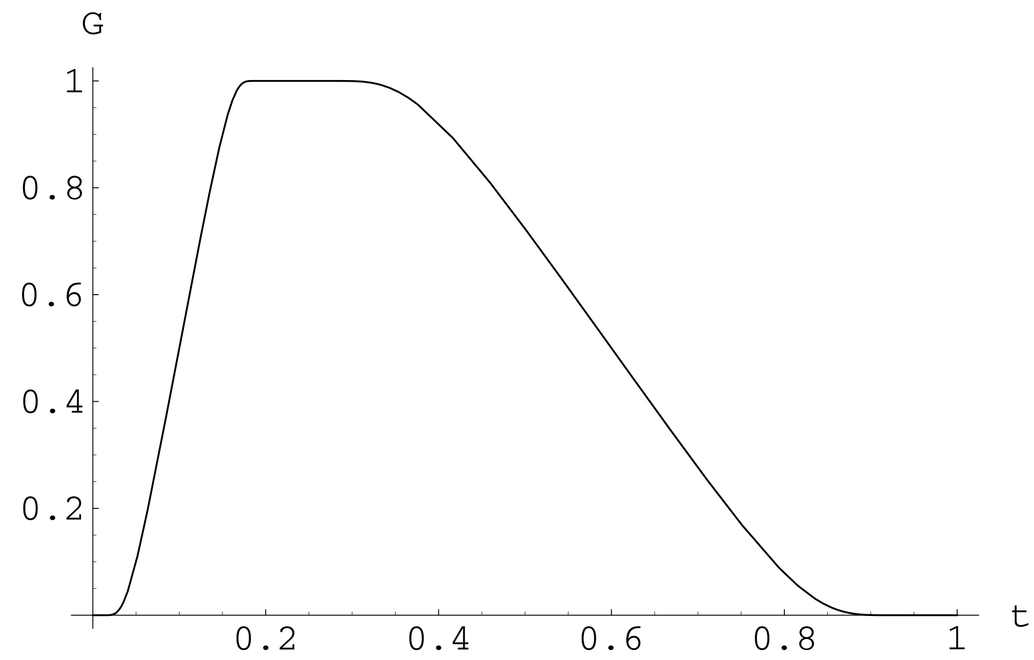

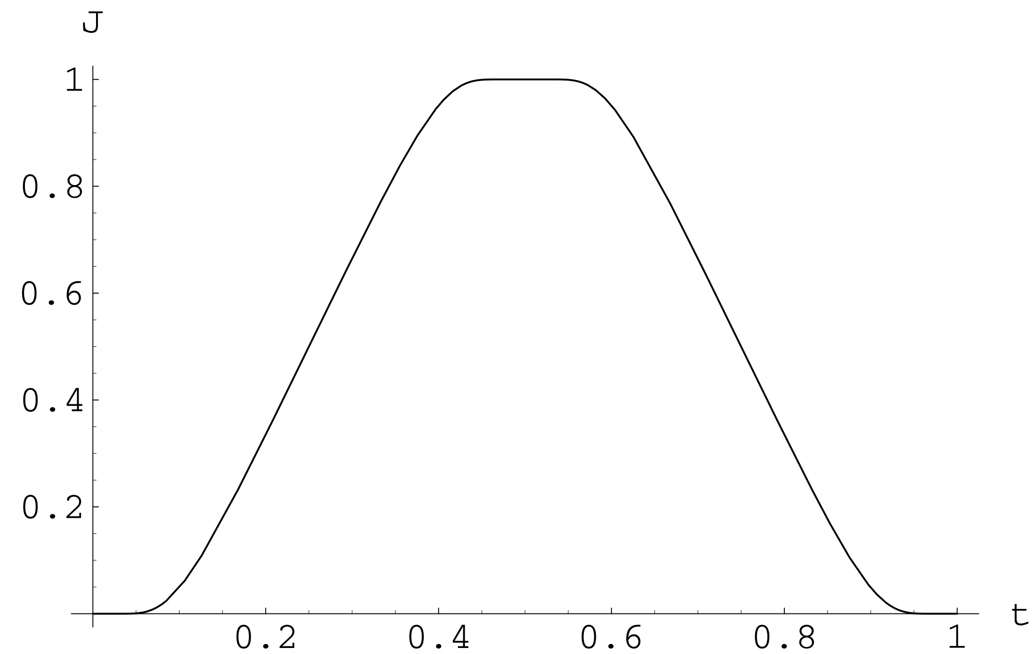

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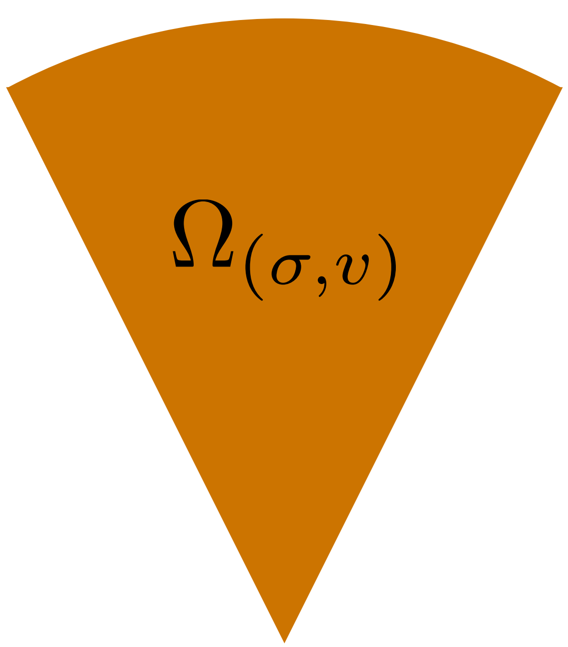

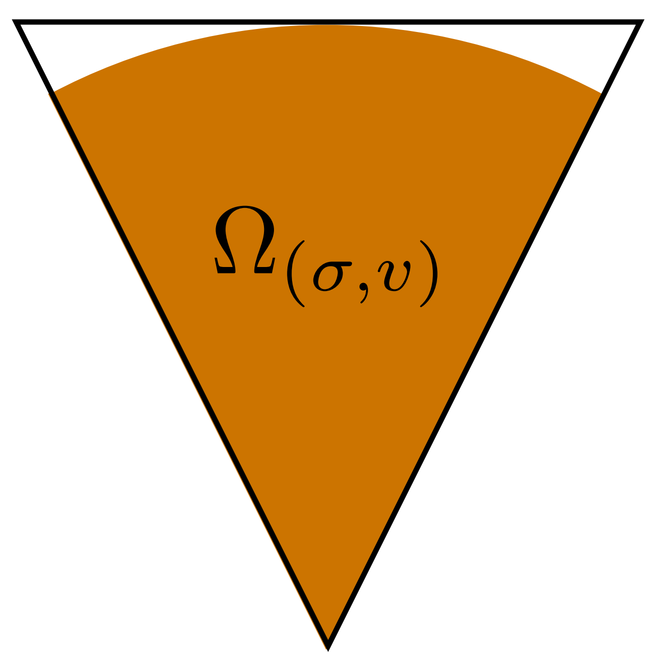

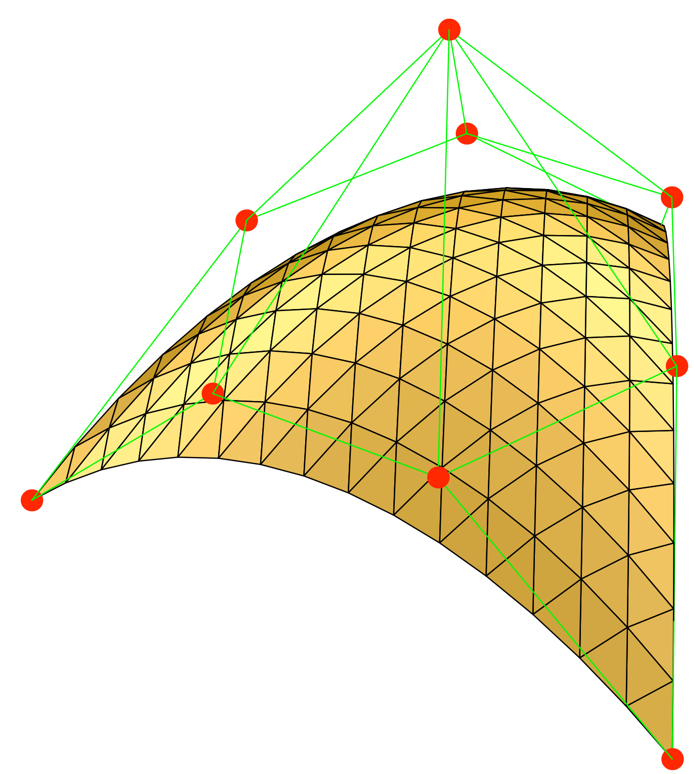

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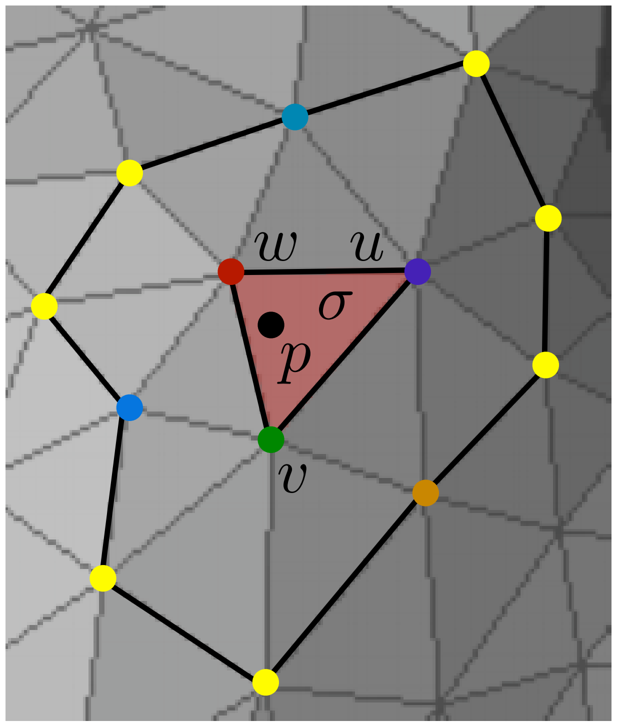

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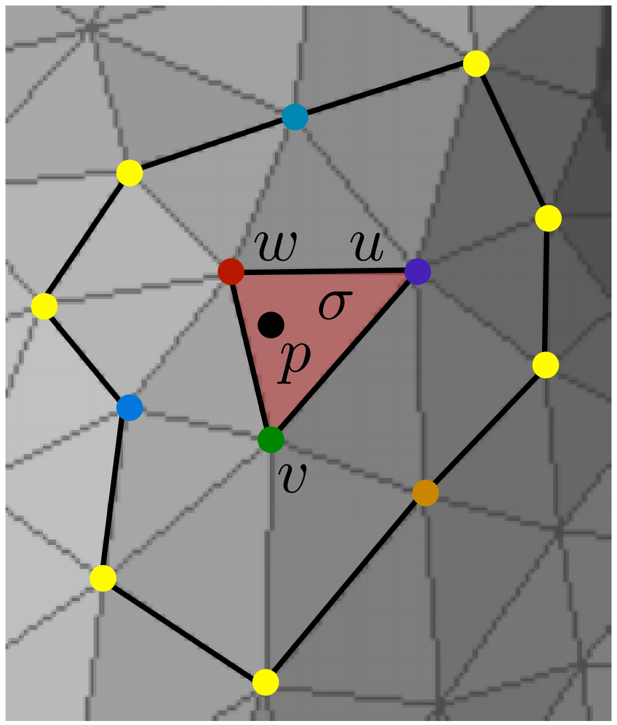

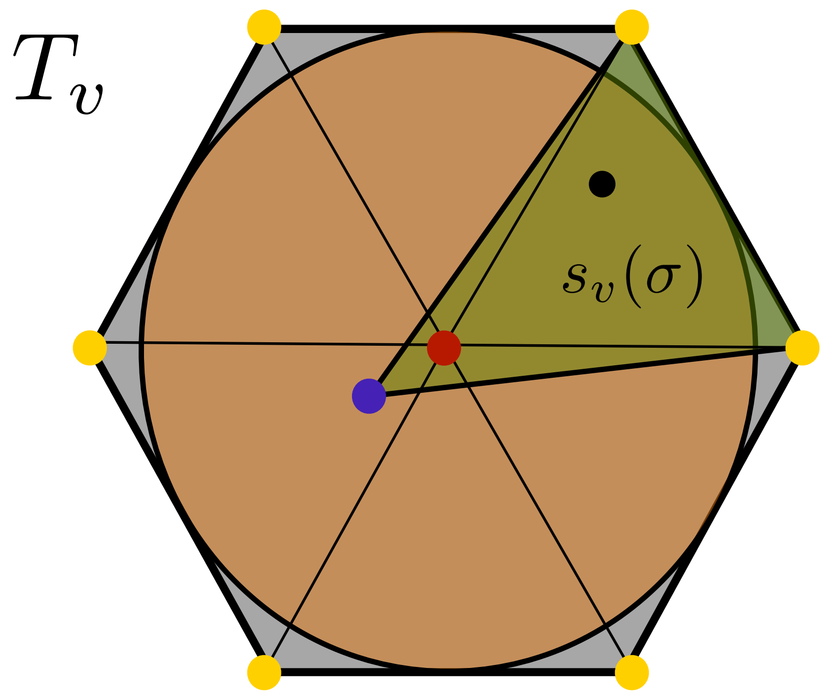

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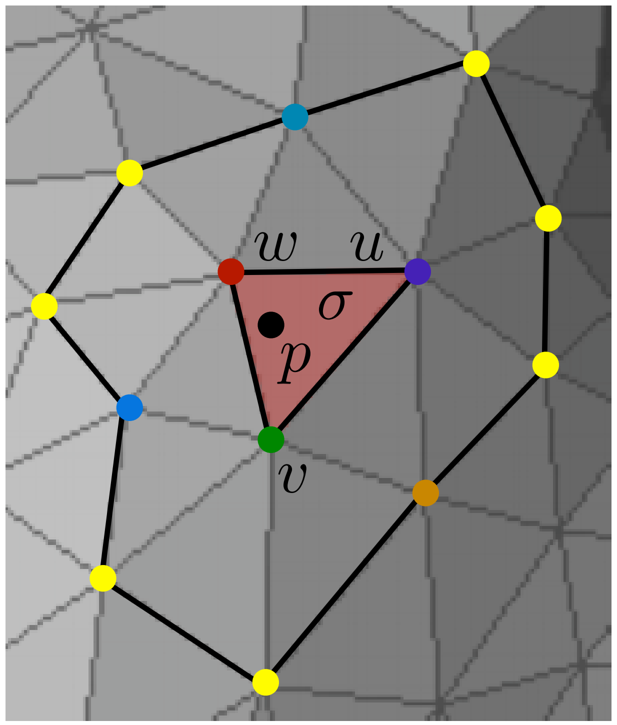

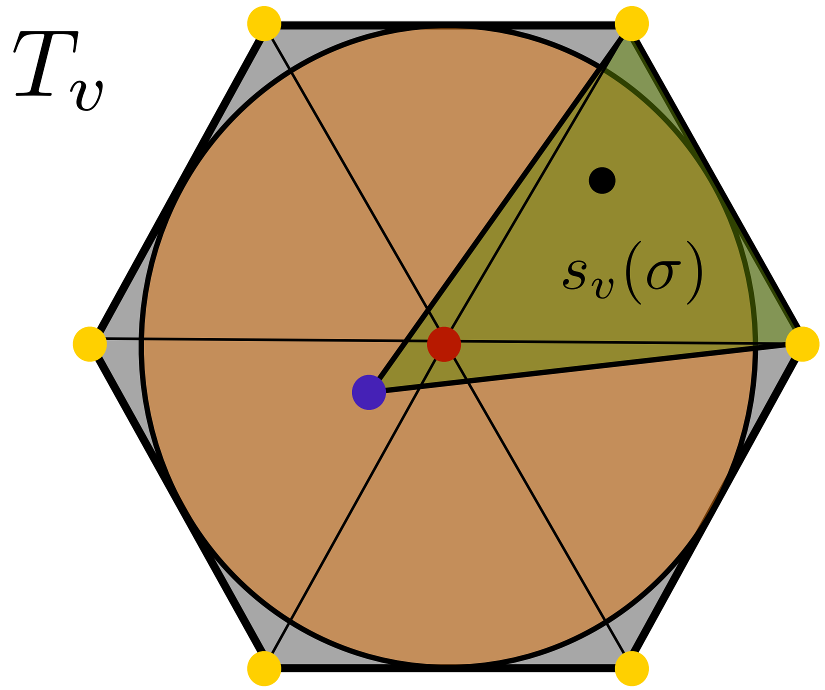

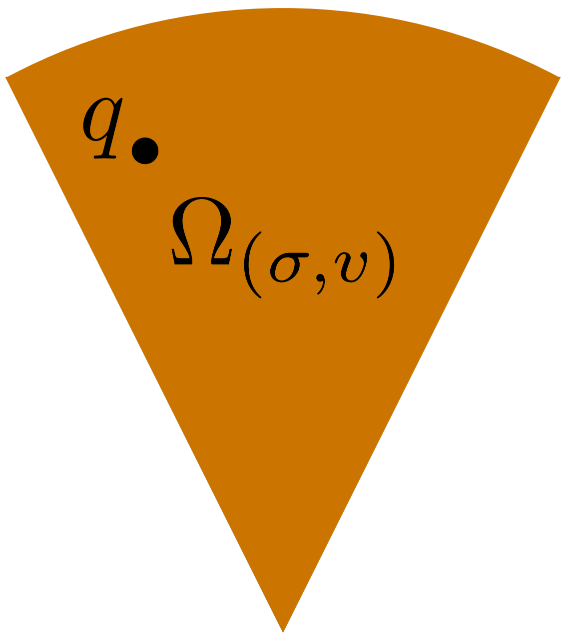

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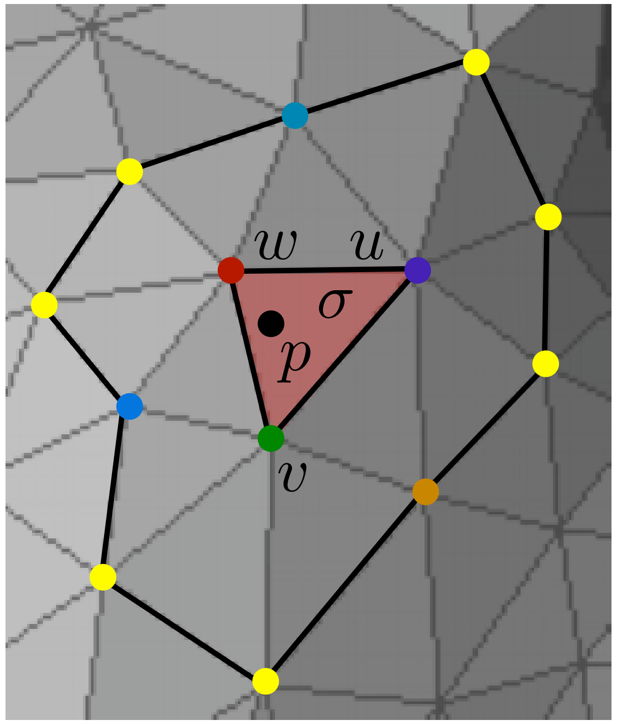
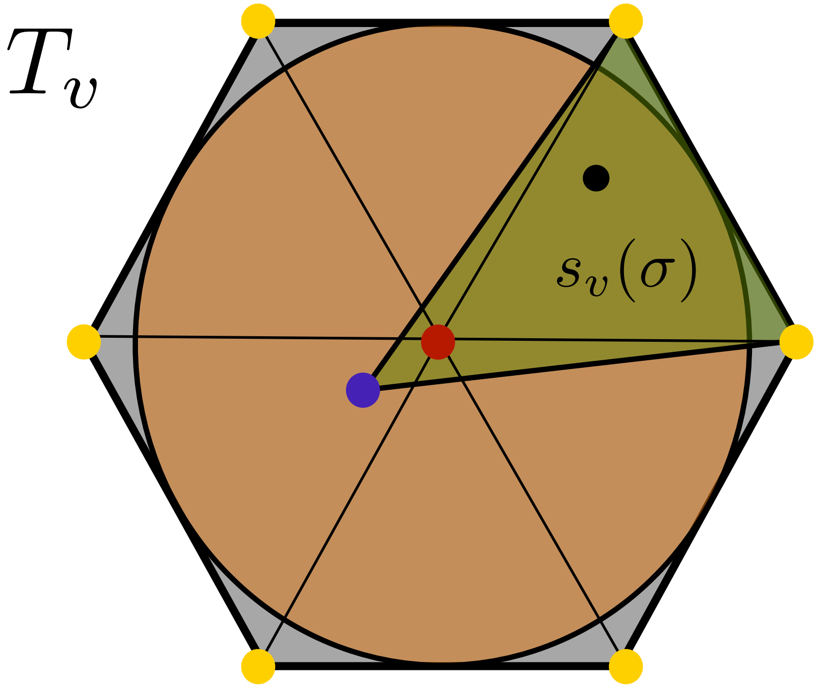
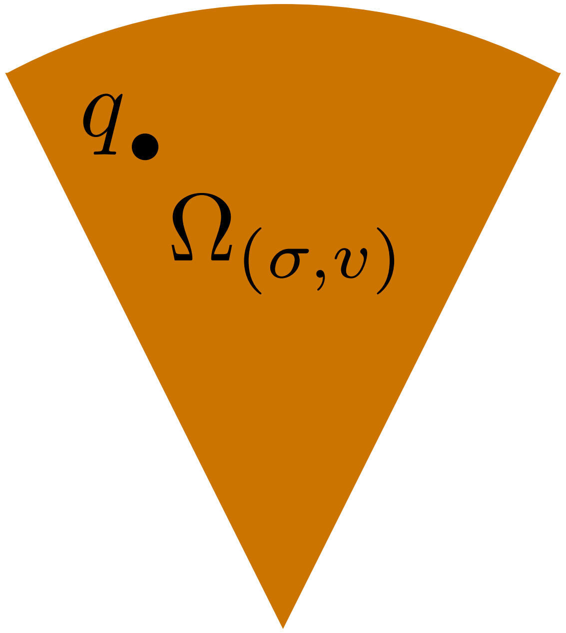
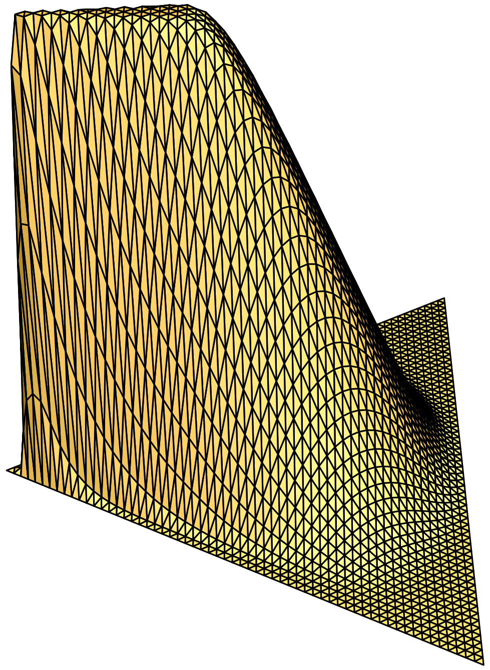
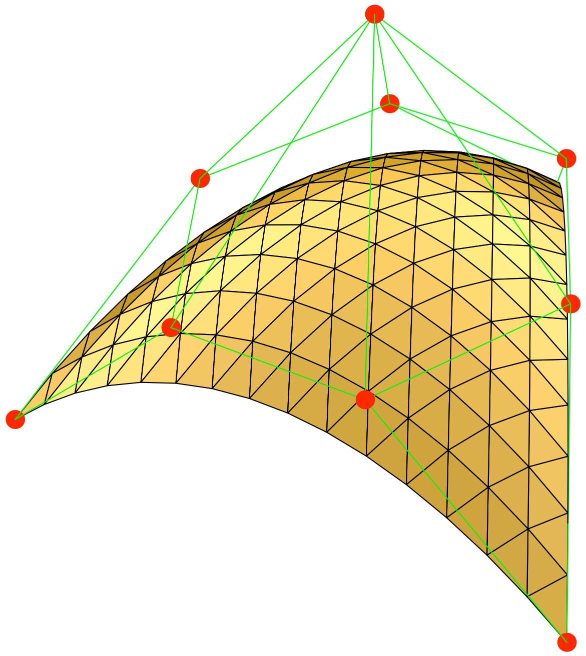


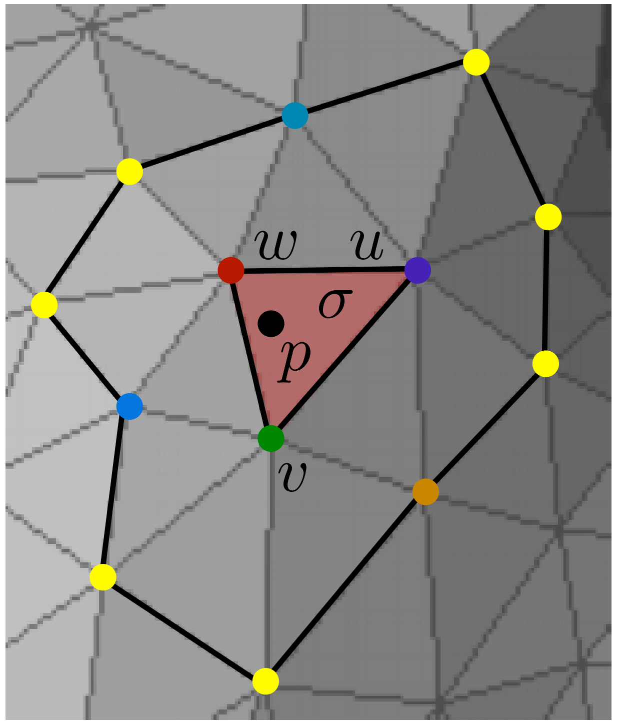


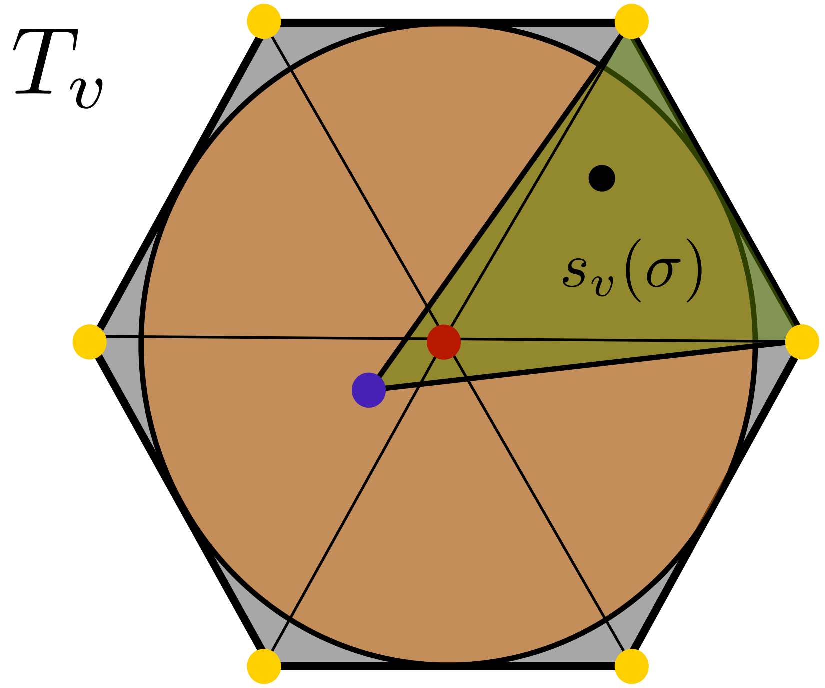


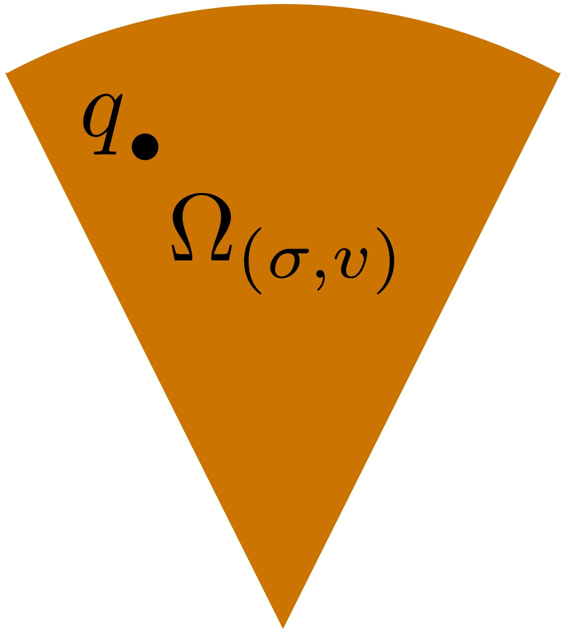


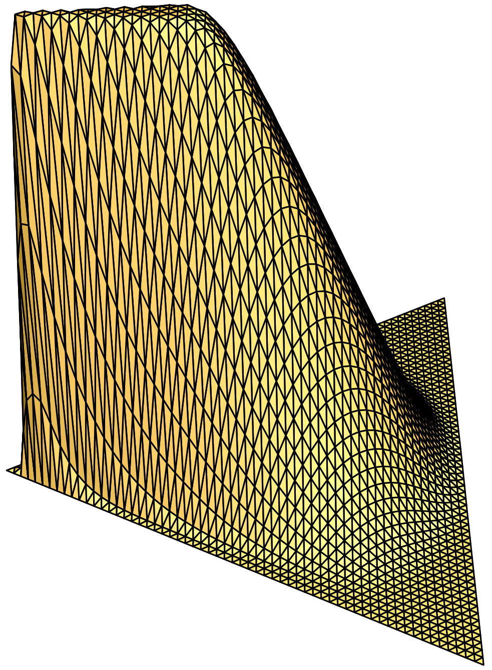


Building Parametrizations



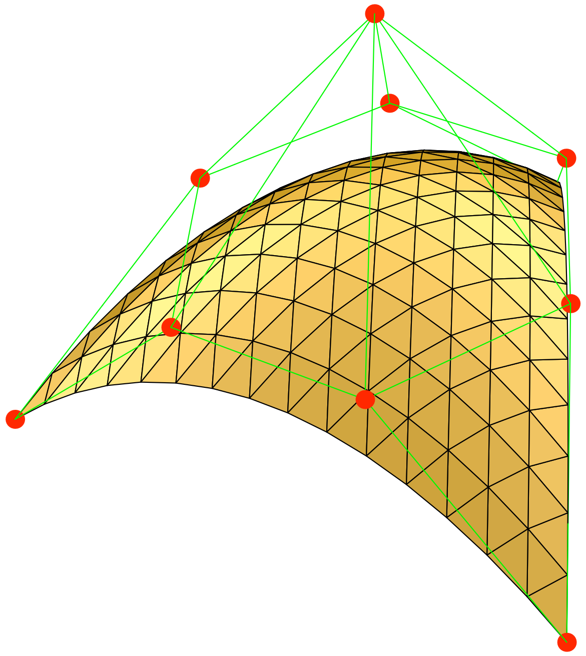






Contribution of $\Omega_{(\sigma,v)}$:

$$\gamma_{(\sigma,v)}(q) \cdot \psi_{(\sigma,v)}(q)$$



Building Parametrizations

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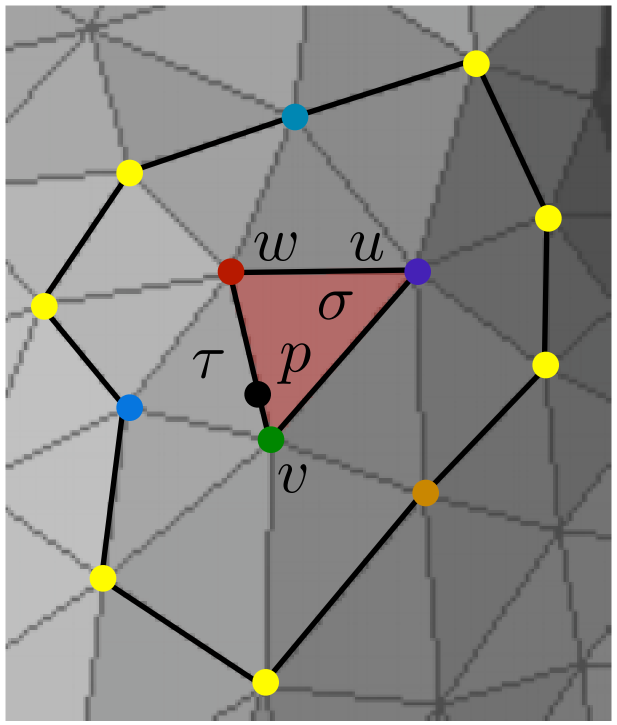

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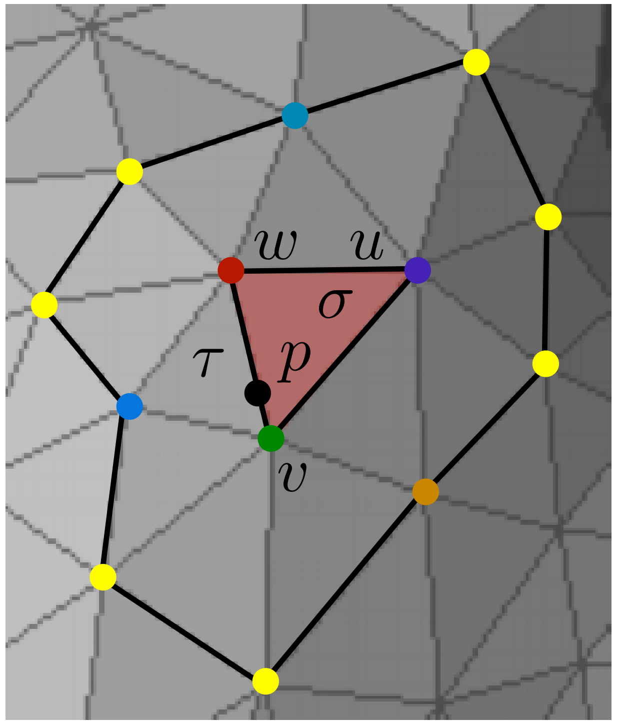

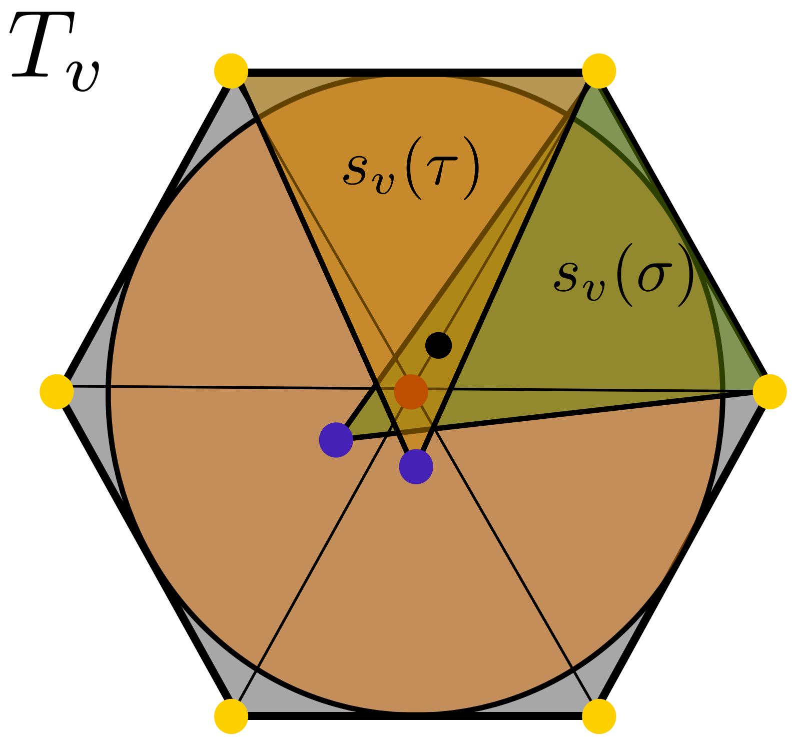

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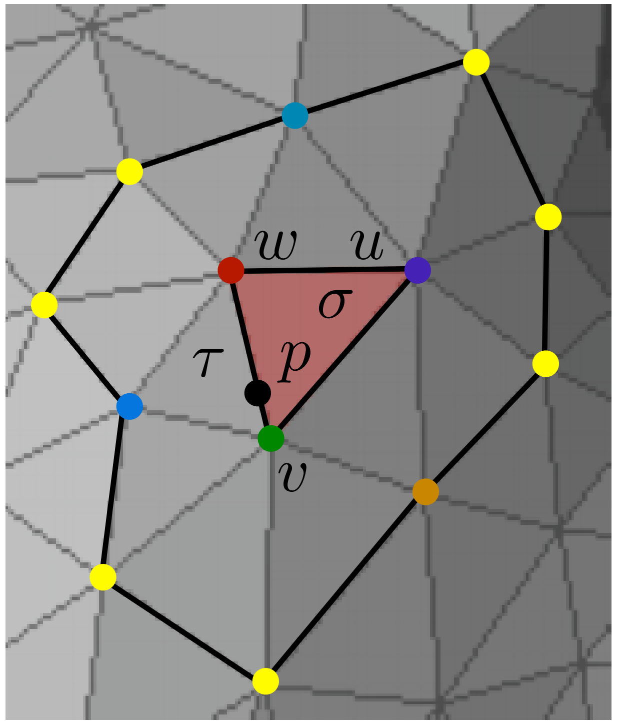

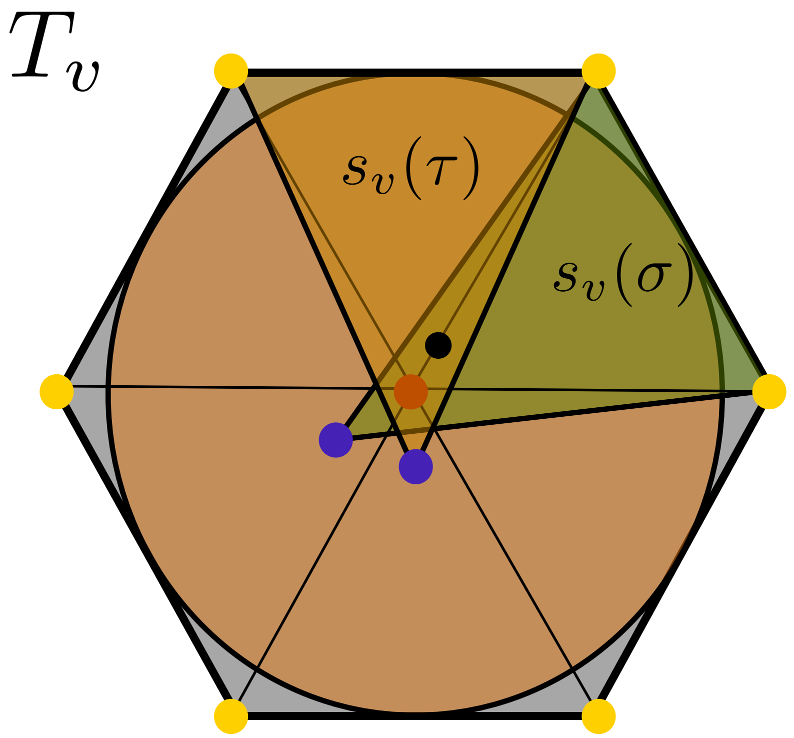

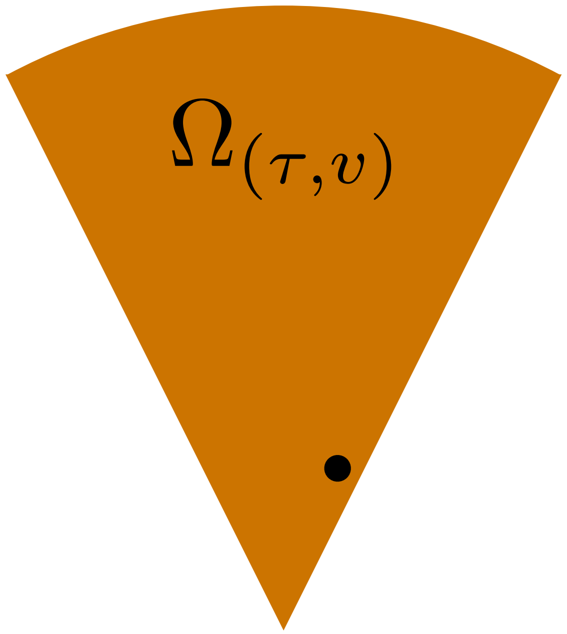

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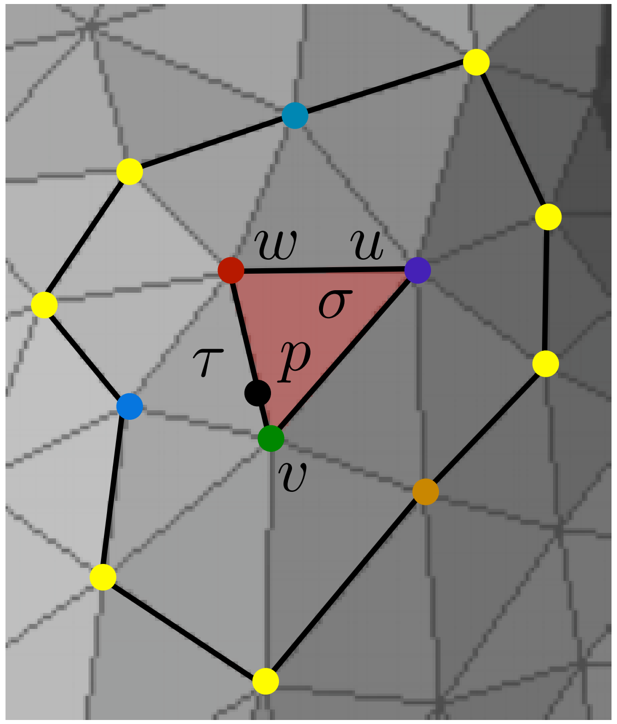

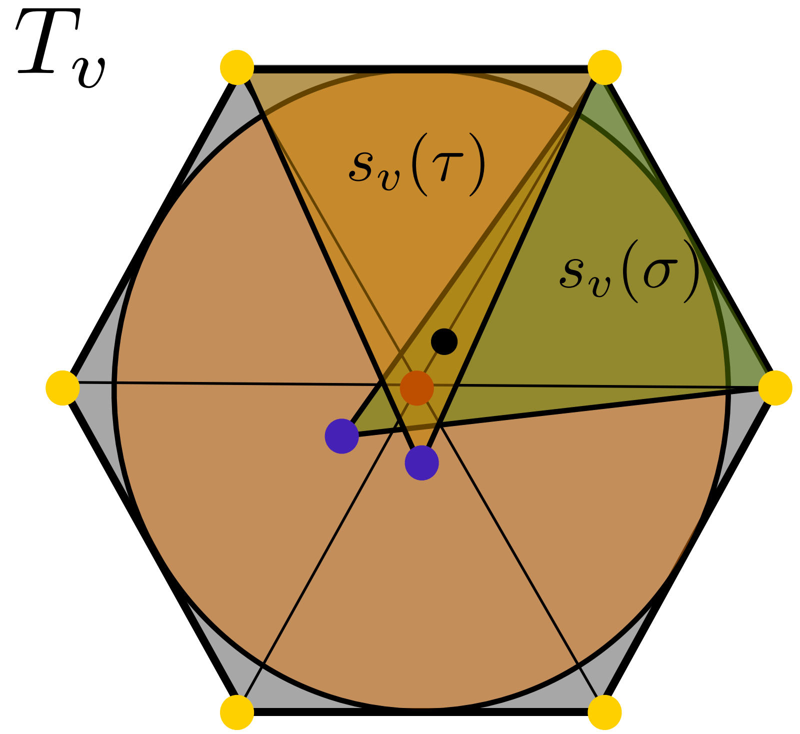

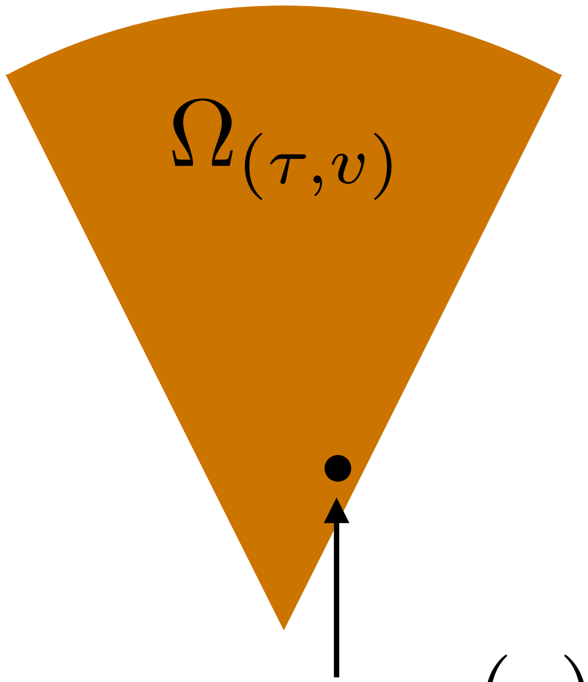

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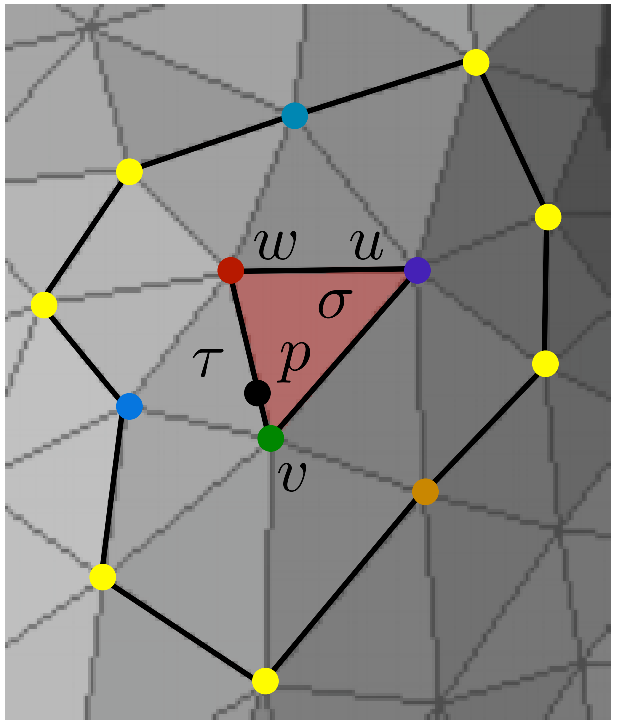

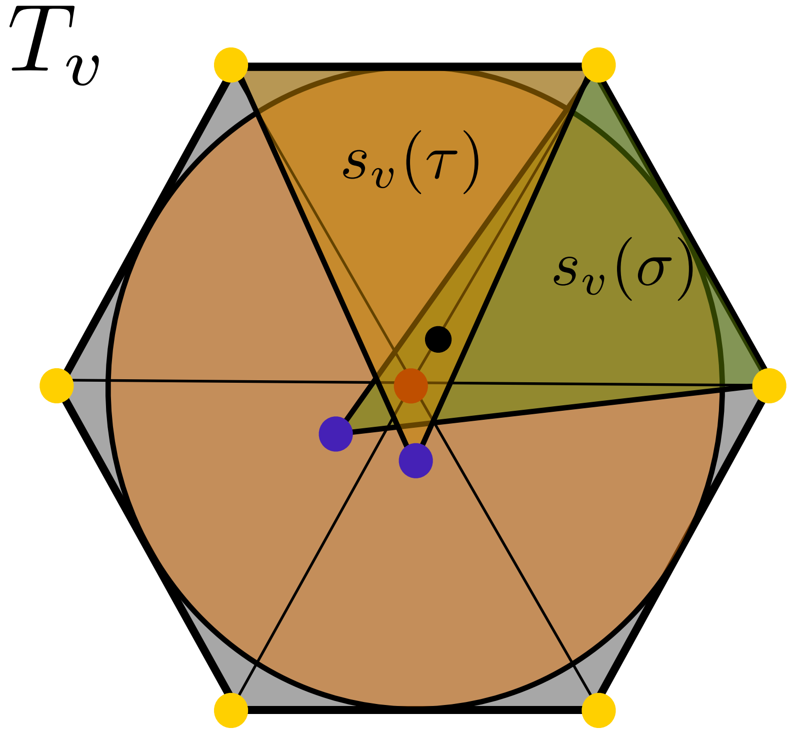

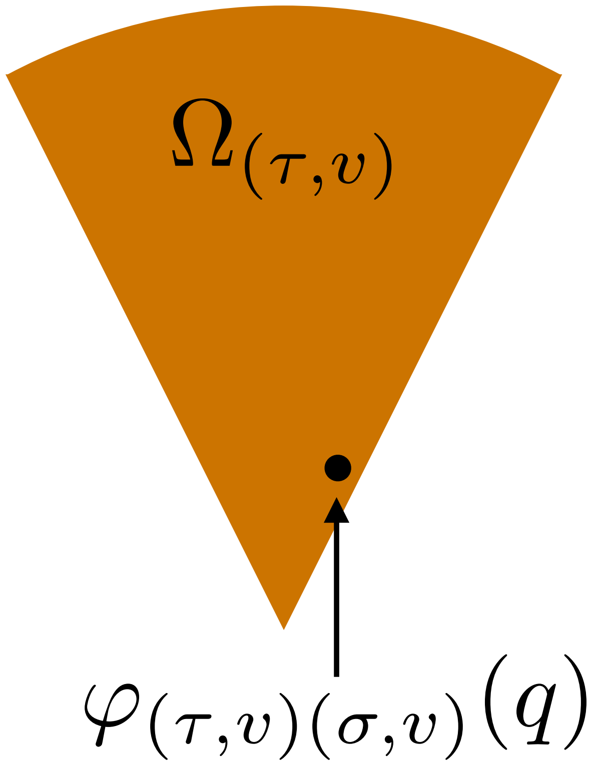

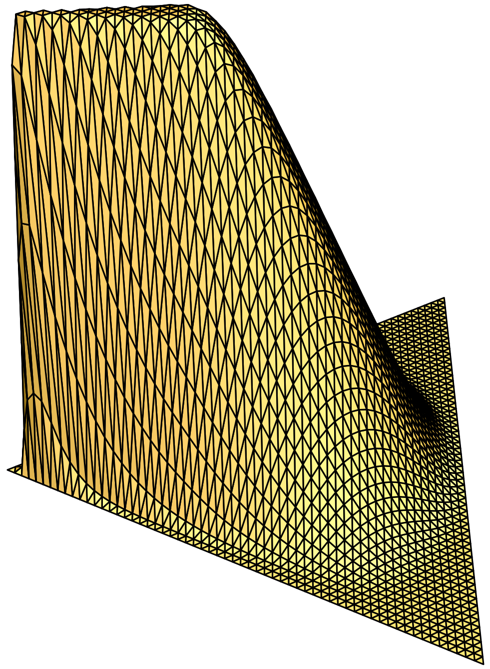

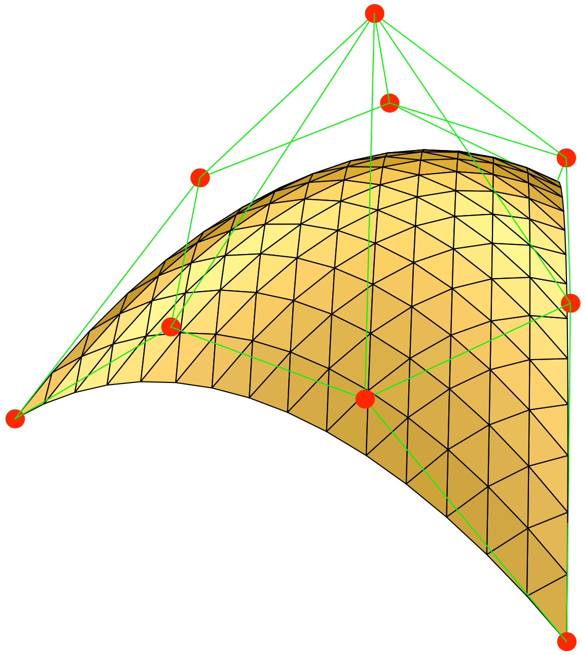

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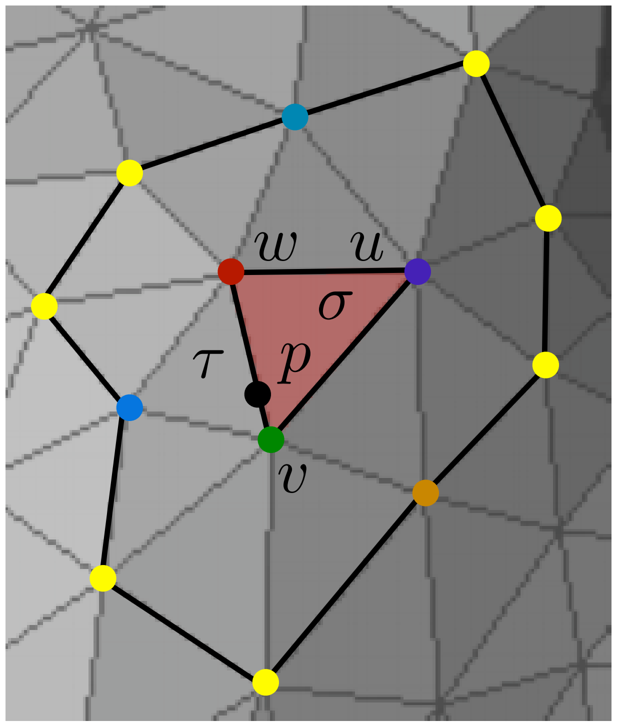

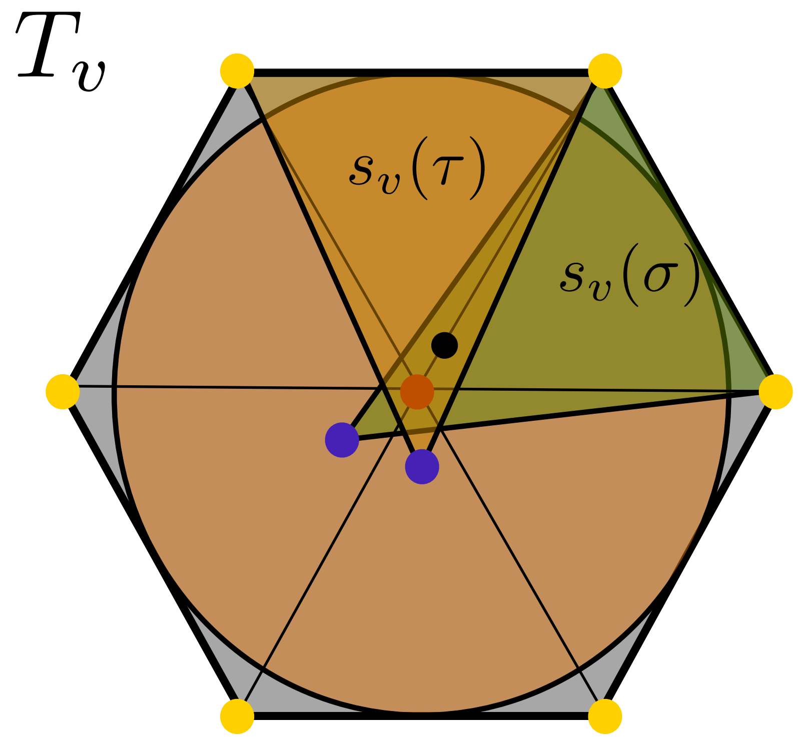

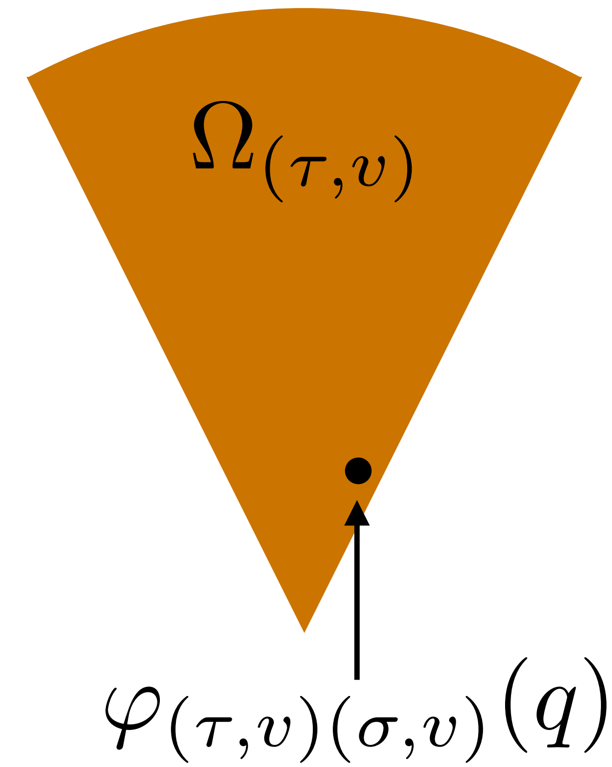

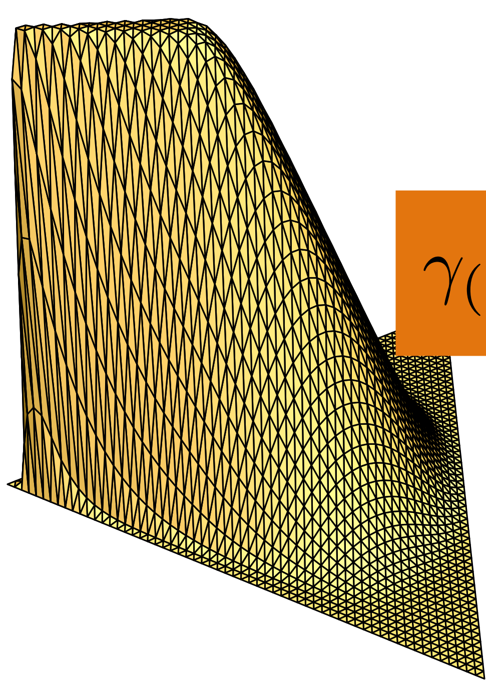

Contribution of $\Omega_{(\tau,v)}$:

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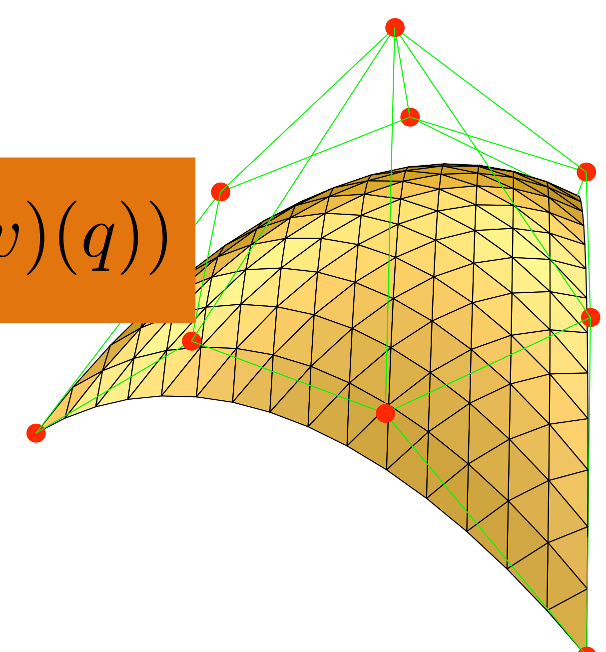

Building Parametrizations

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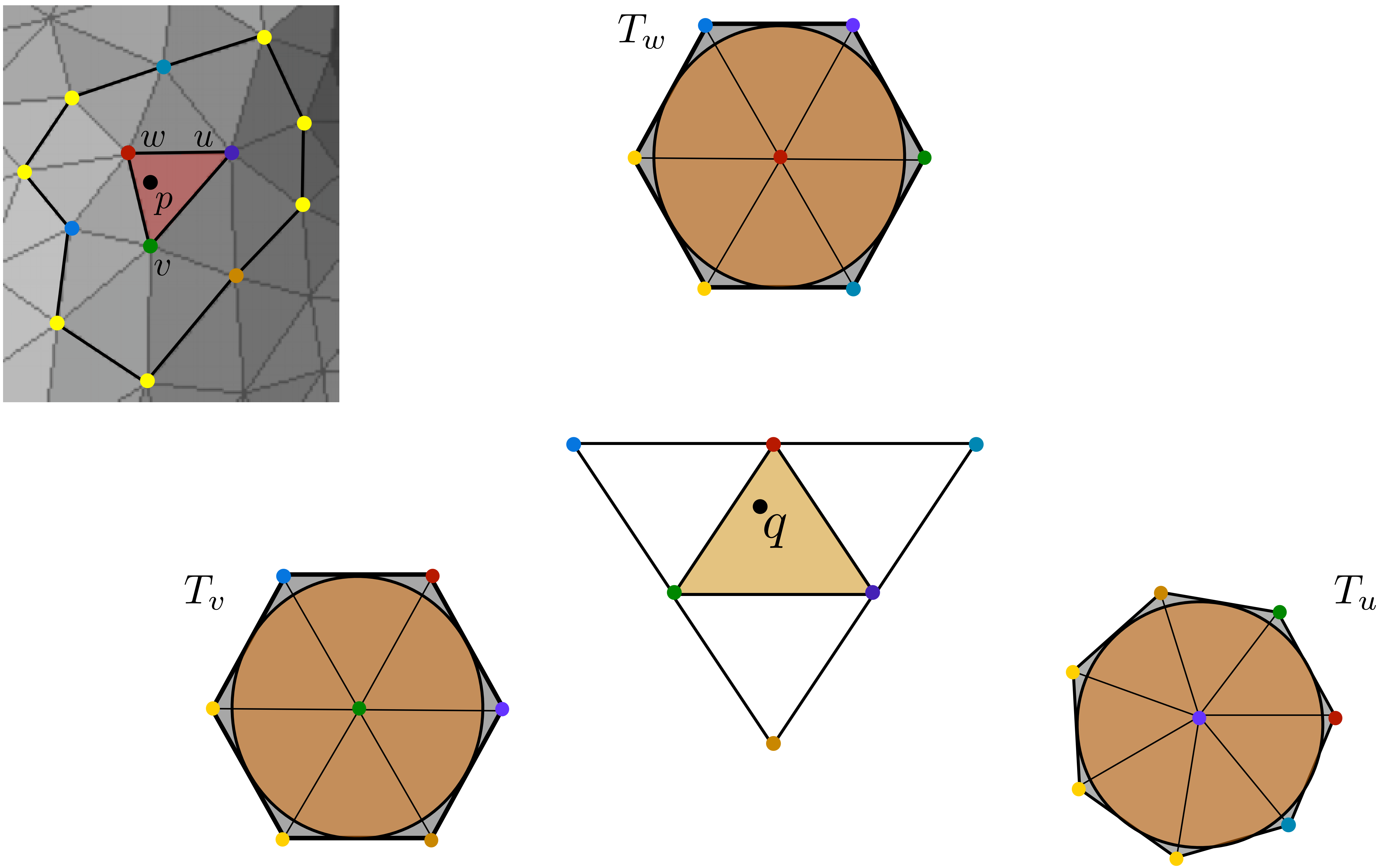

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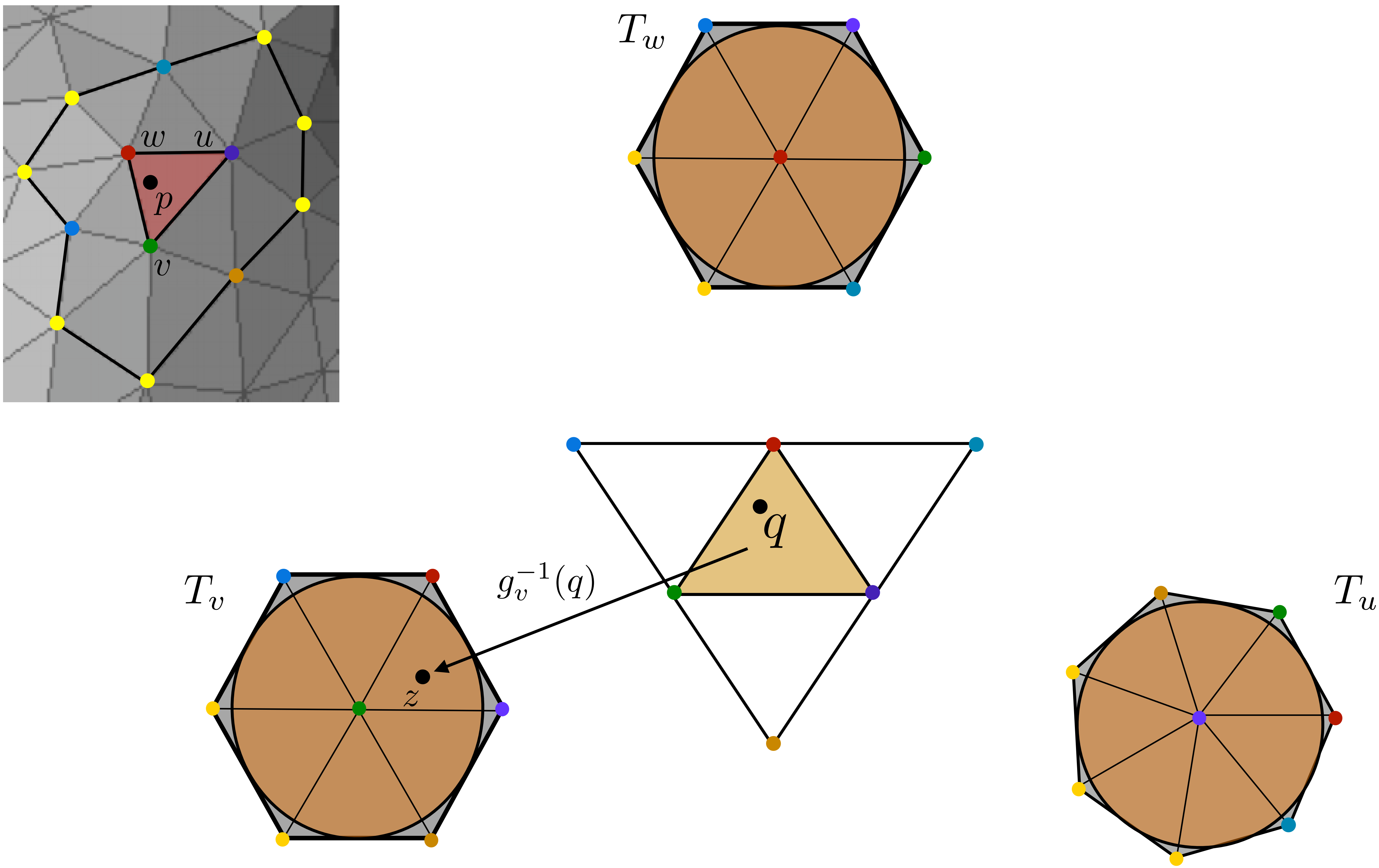

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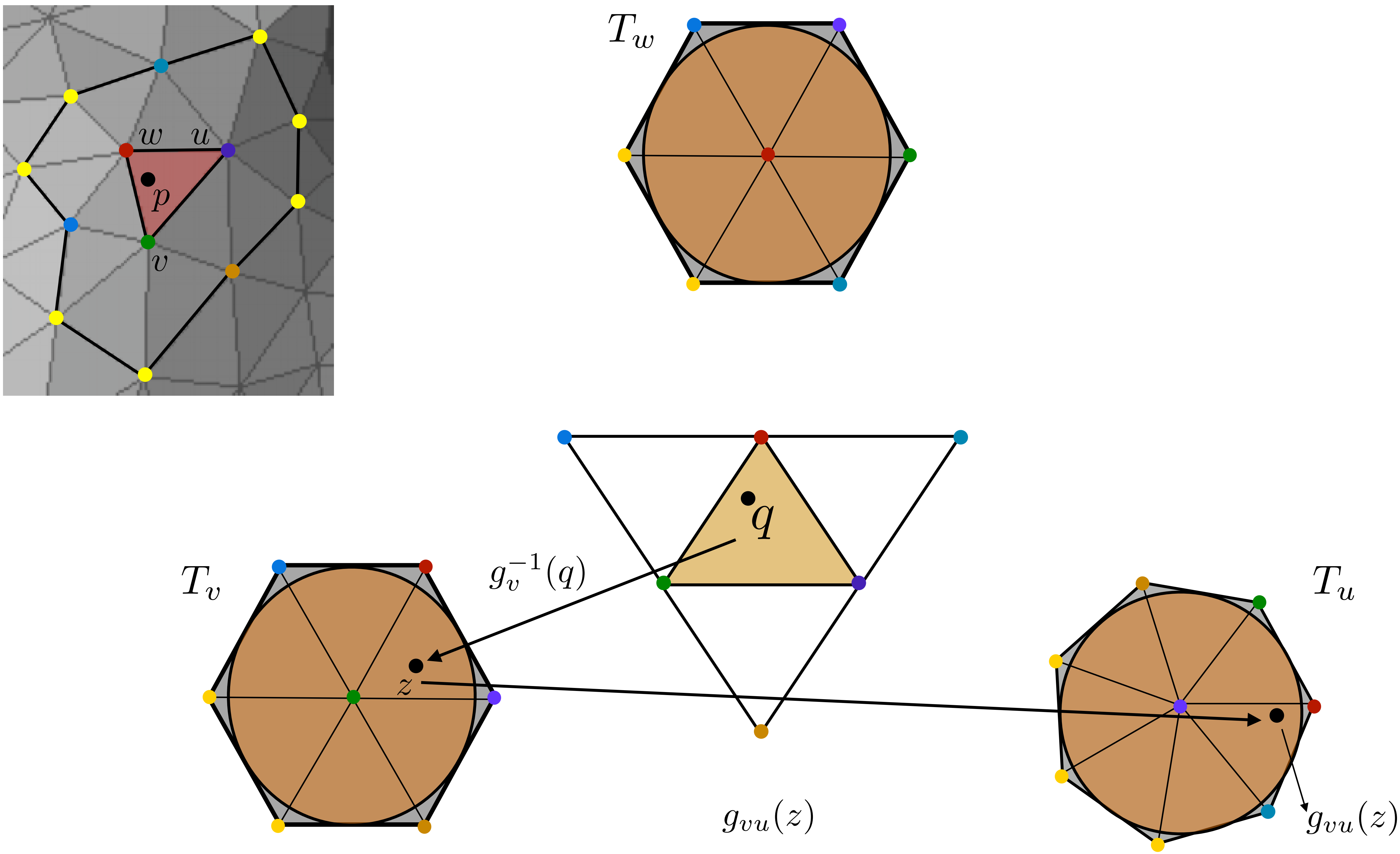

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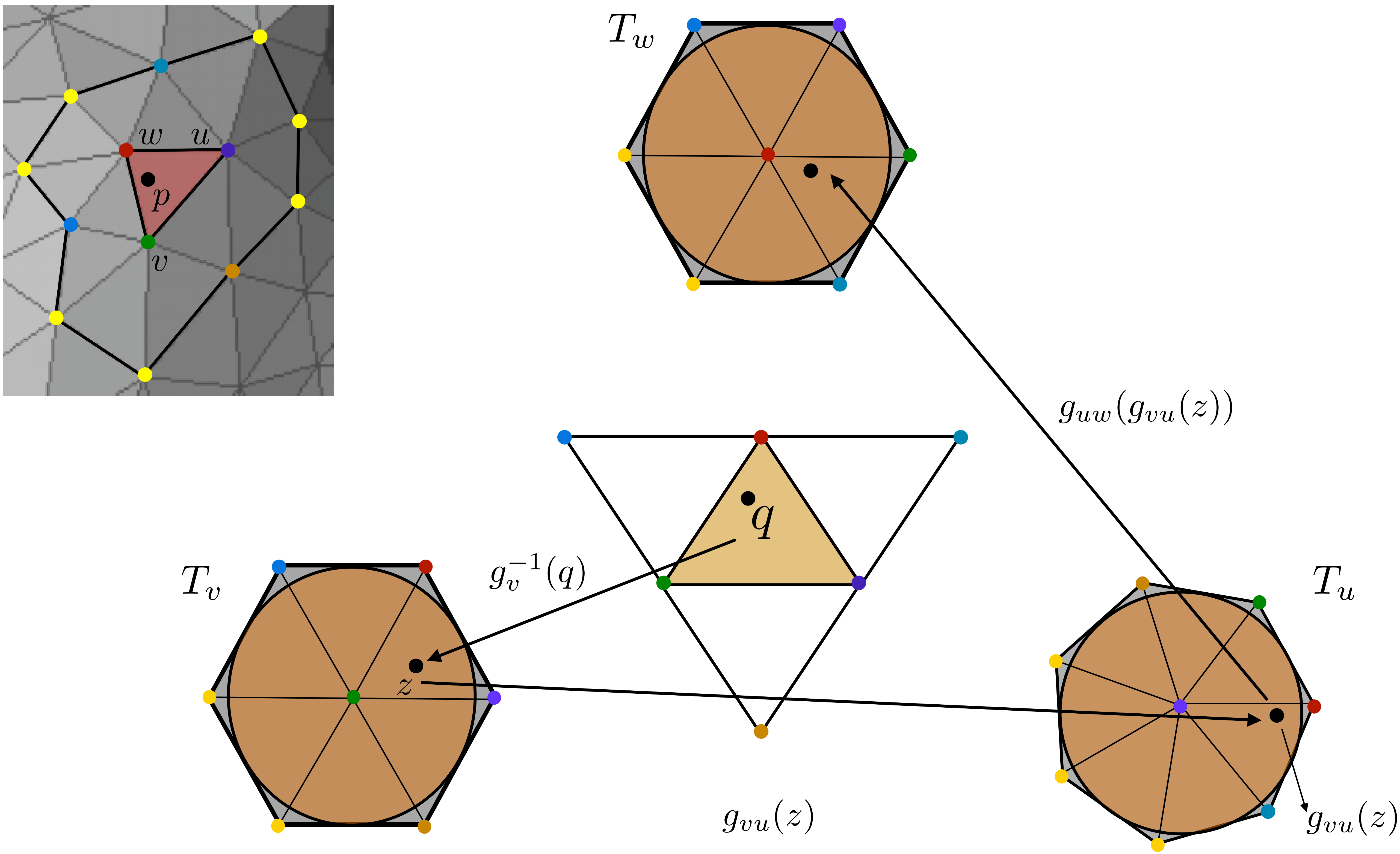


Building Parametrizations

Building Parametrizations

For each $(\sigma, v) \in I$, we define a **parametrization**,

$$\theta_{(\sigma,v)} : \Omega_{(\sigma,v)} \to \mathbb{R}^3 \,,$$

such that for every $p \in \Omega_{(\sigma,v)}$,

$$\theta_{(\sigma,v)}(p) = \sum_{(\tau,u)\in J(p)} \nu_{(\tau,u)}(p) \cdot \psi_{(\tau,u)}(\varphi_{(\tau,u)(\sigma,v)}(p)) \,,$$

where

Building Parametrizations

Building Parametrizations

$$\nu_{(\tau,u)}(p) = \frac{\gamma_{(\tau,u)}(\varphi_{(\tau,u)(\sigma,v)}(p))}{\displaystyle\sum_{(\eta,w)\in J(p)} \gamma_{(\eta,w)}(\varphi_{(\eta,w)(\sigma,v)}(p))}$$

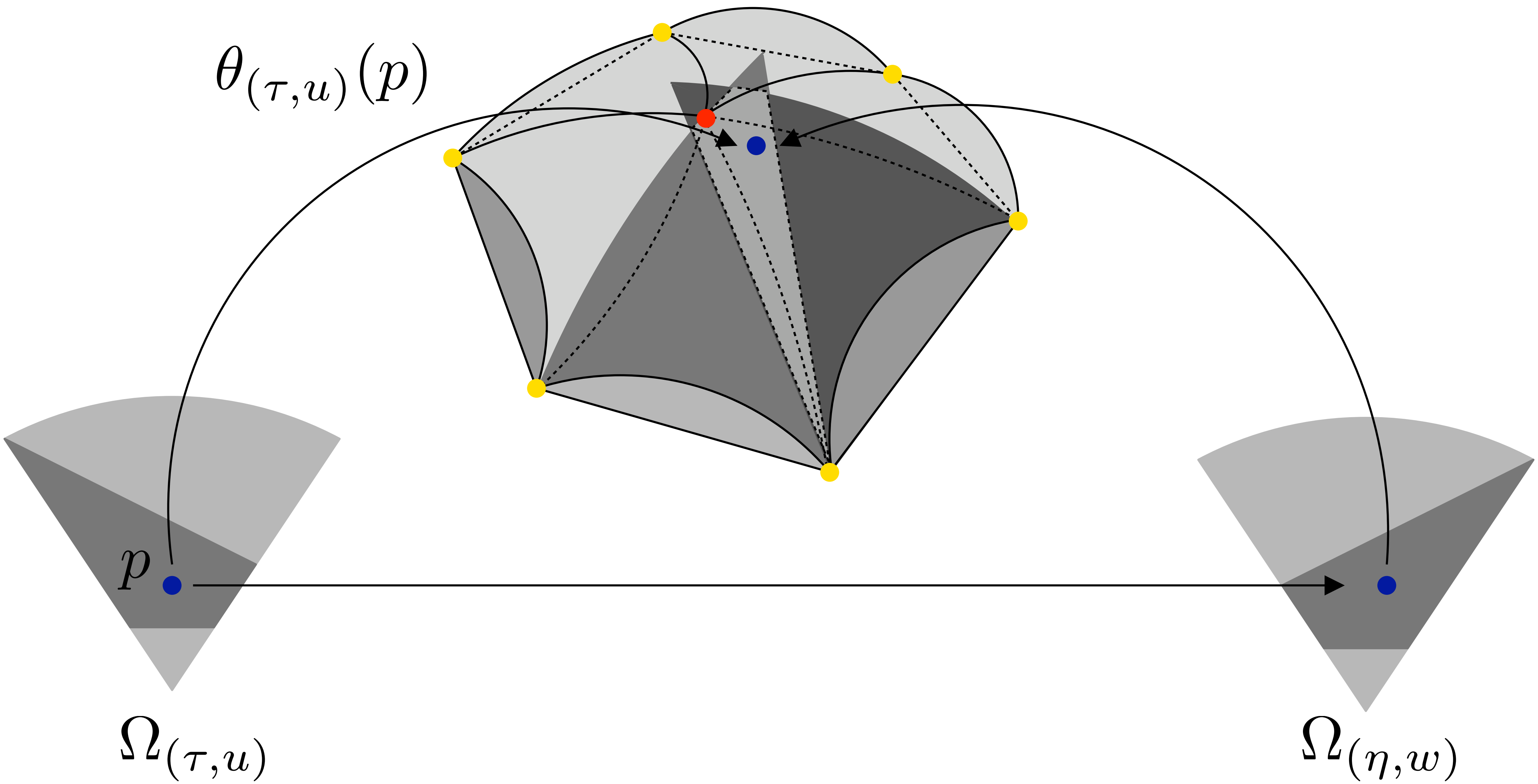
and

$$J(p) = \left\{(\eta, w) \in I \mid p \in \Omega_{(\sigma,v)(\eta,w)}\right\}.$$

Building Parametrizations

Building Parametrizations

Parametrizations are consistent!



Building Parametrizations

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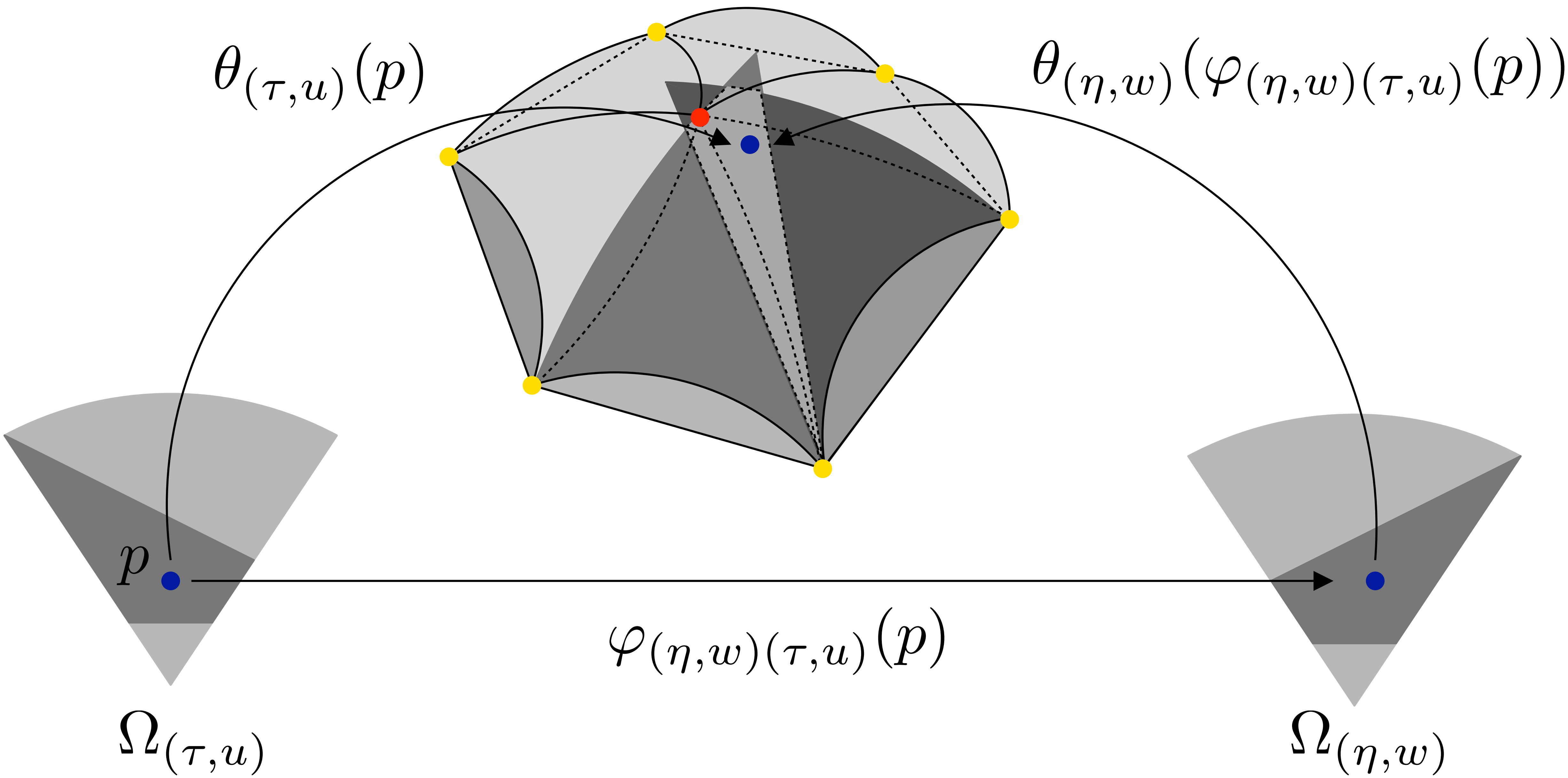
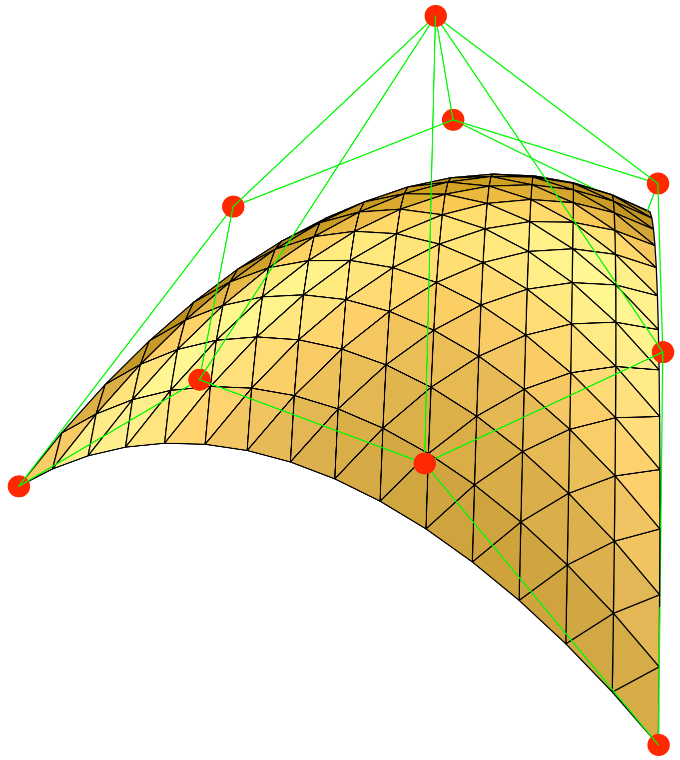


Building Parametrizations

$$\psi_{(\tau,u)}(\Omega_{(\tau,u)})$$

Building Parametrizations

The control points of $\psi_{(\tau,u)}$ are the solutions of a least squares problem.



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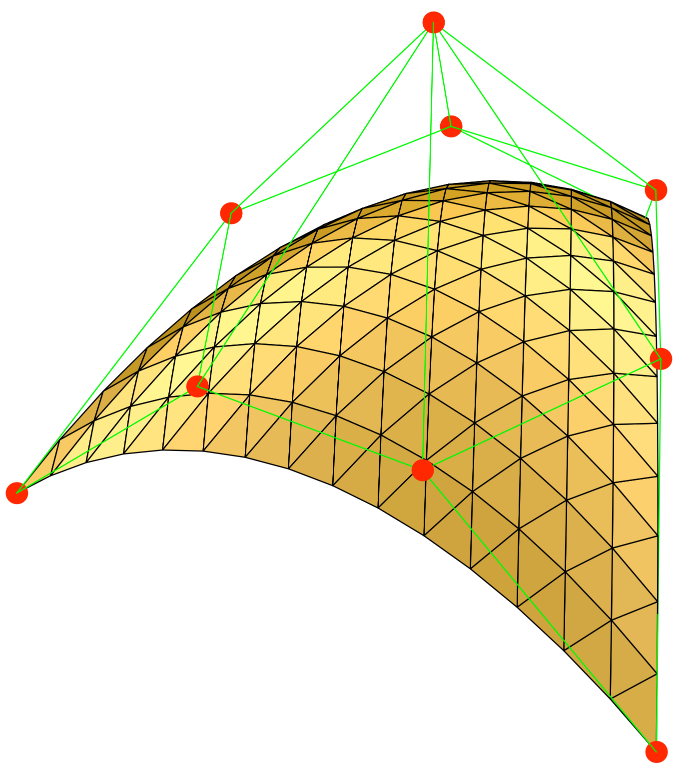
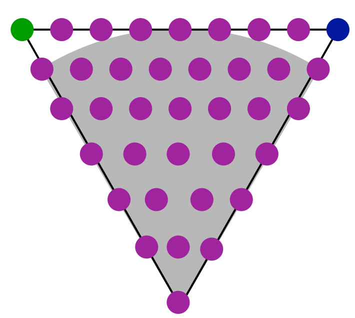
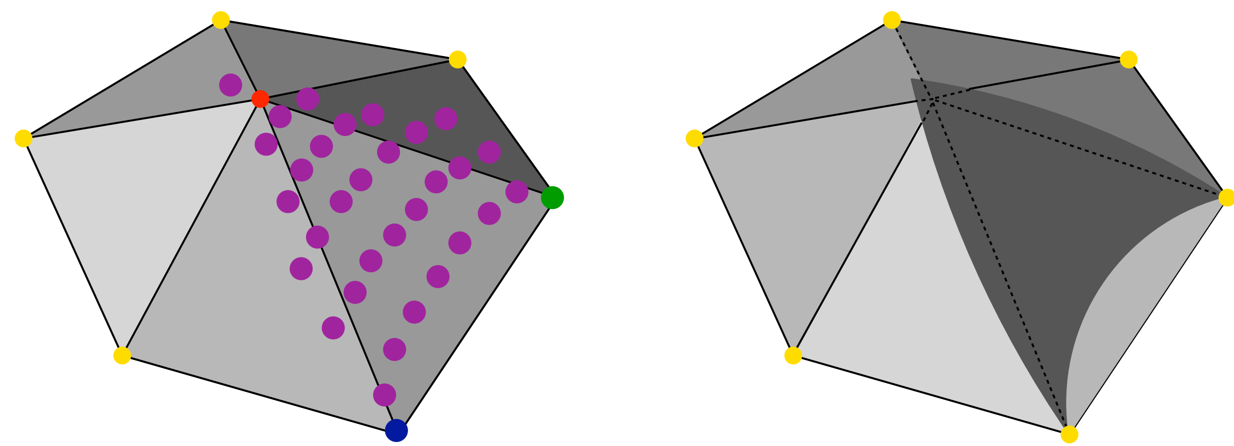




$\Omega_{(\tau,u)}$



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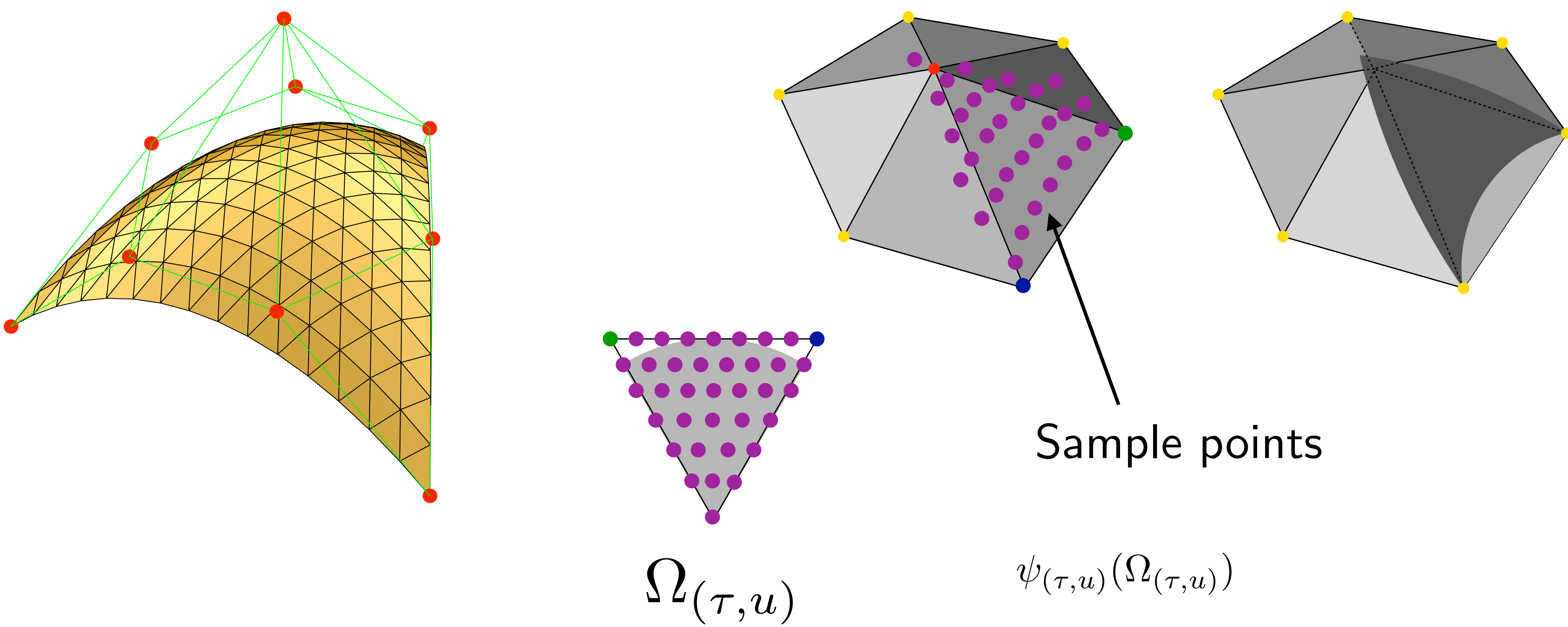

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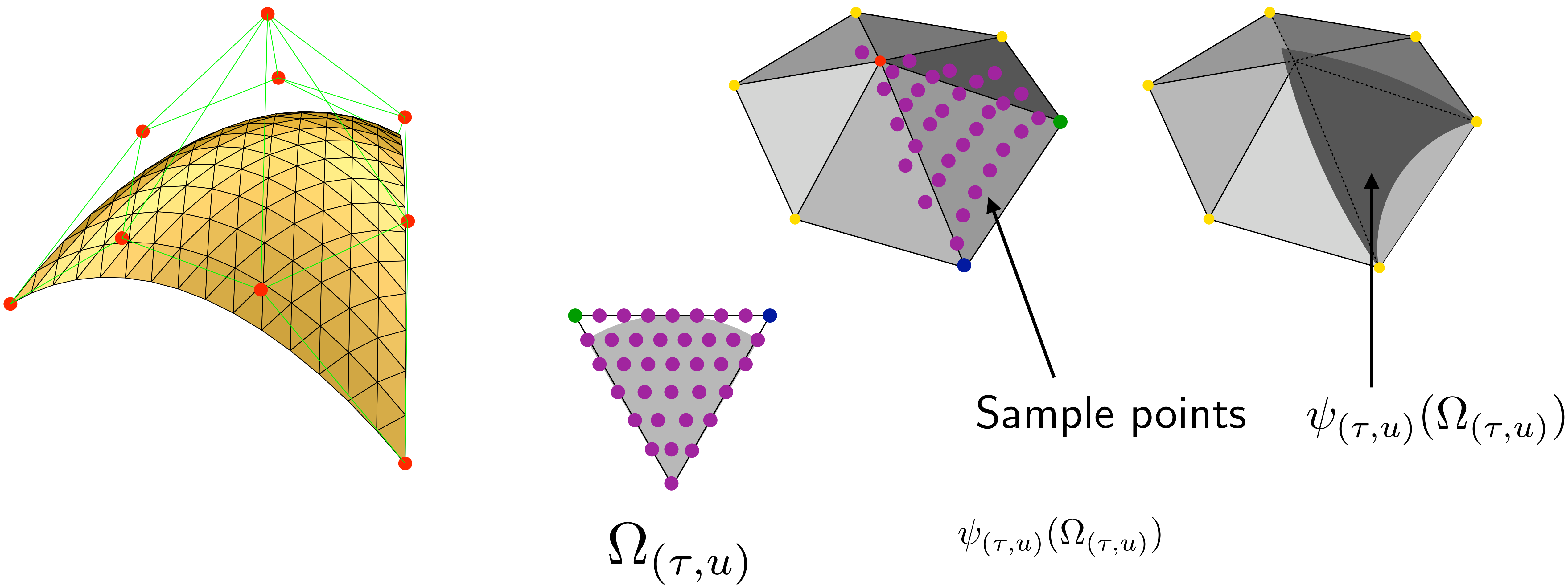

Building Parametrizations

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How can we find the sample points to start with?

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Fit a “curved” surface, S', to S_T and then sample it!

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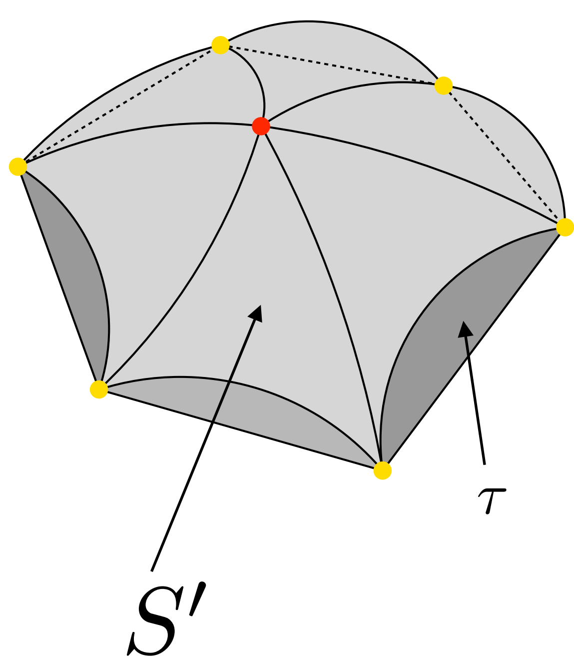

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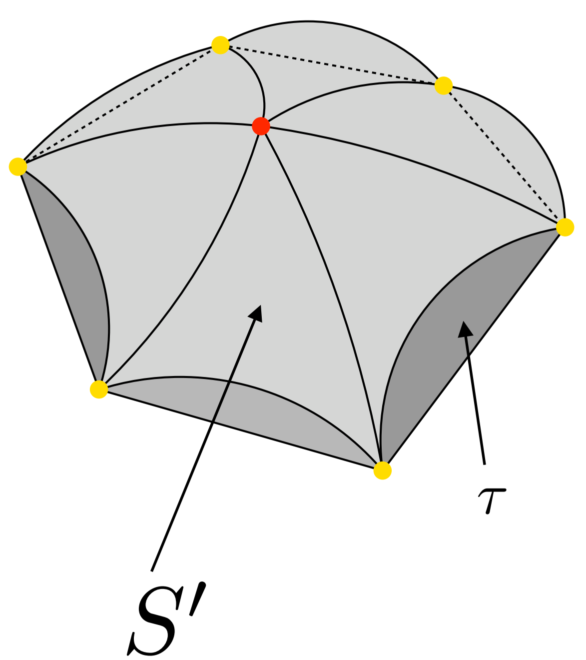

Good choices:

- PN triangle surfaces
- Subdivision surfaces

Building Parametrizations

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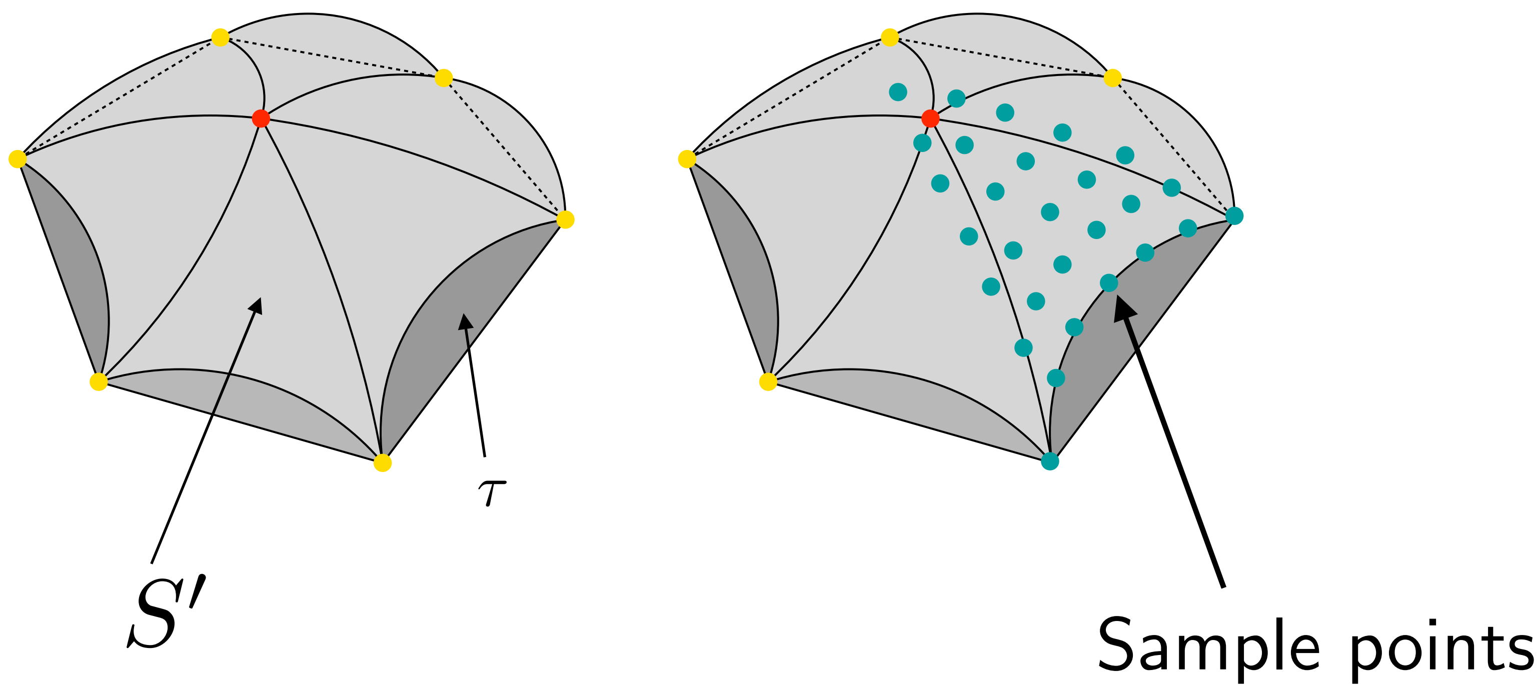

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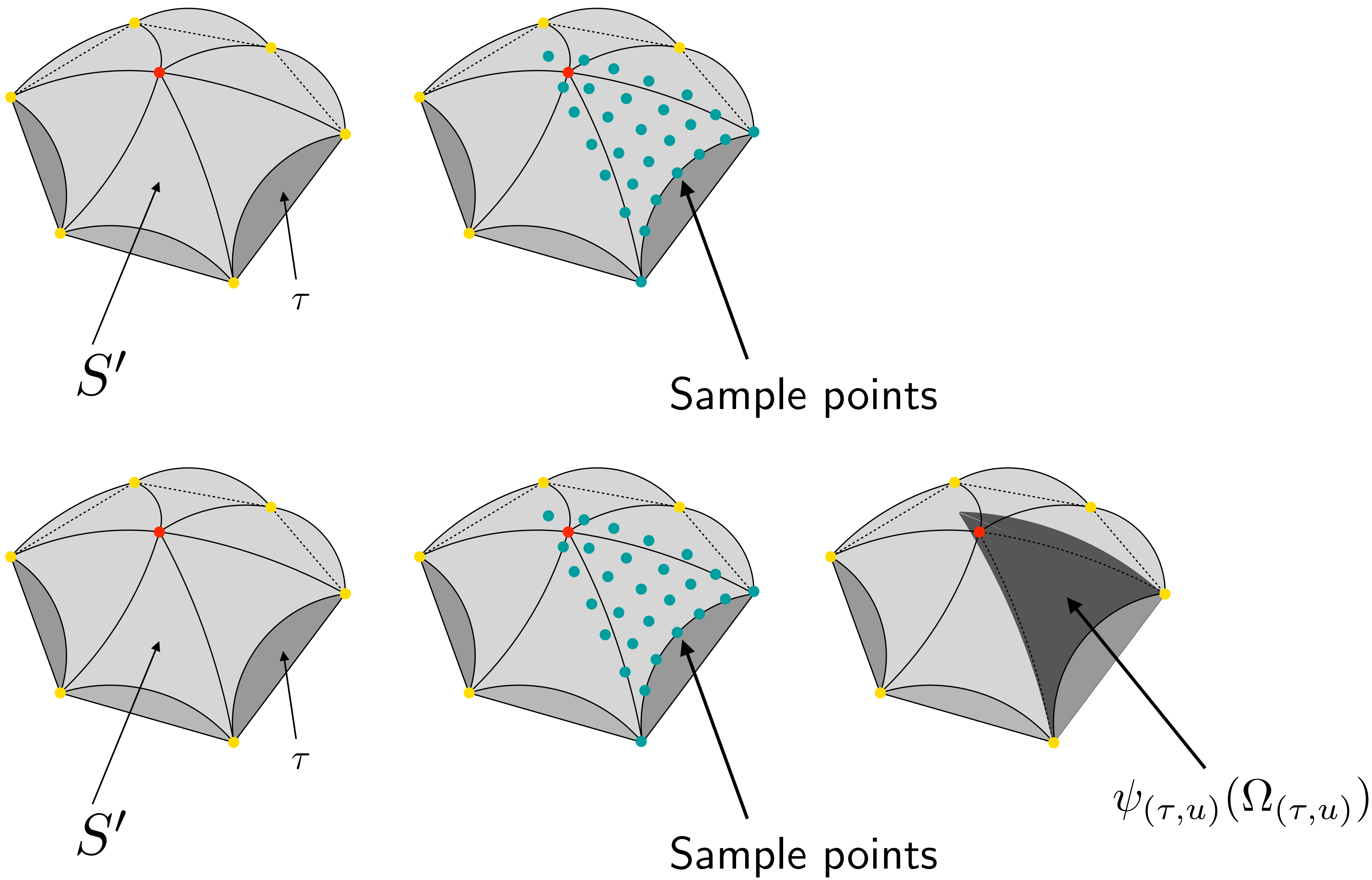
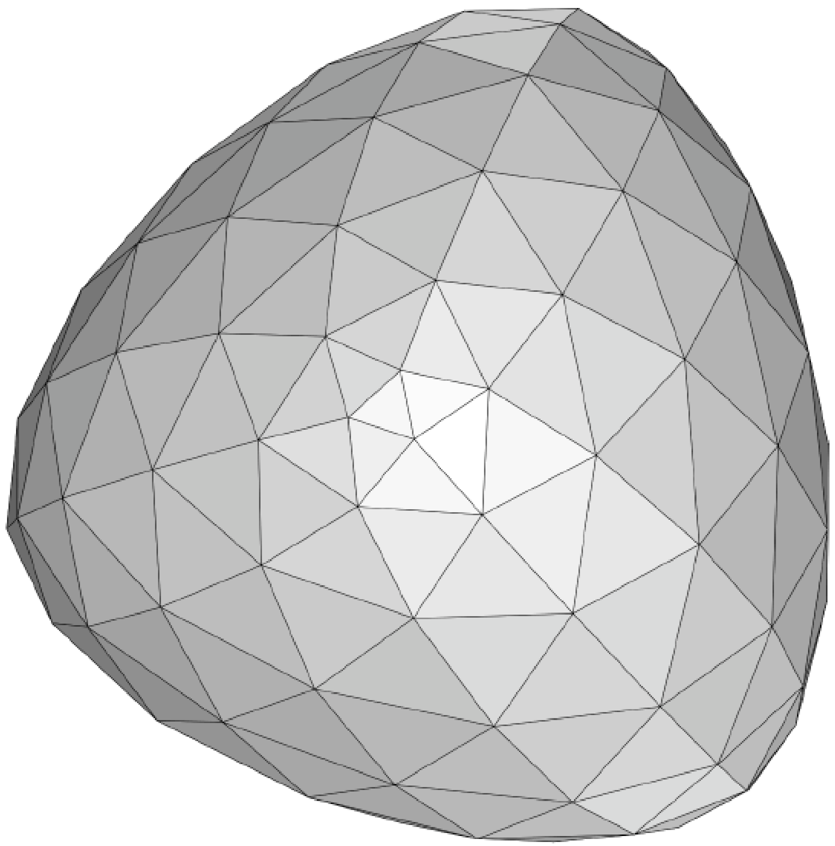


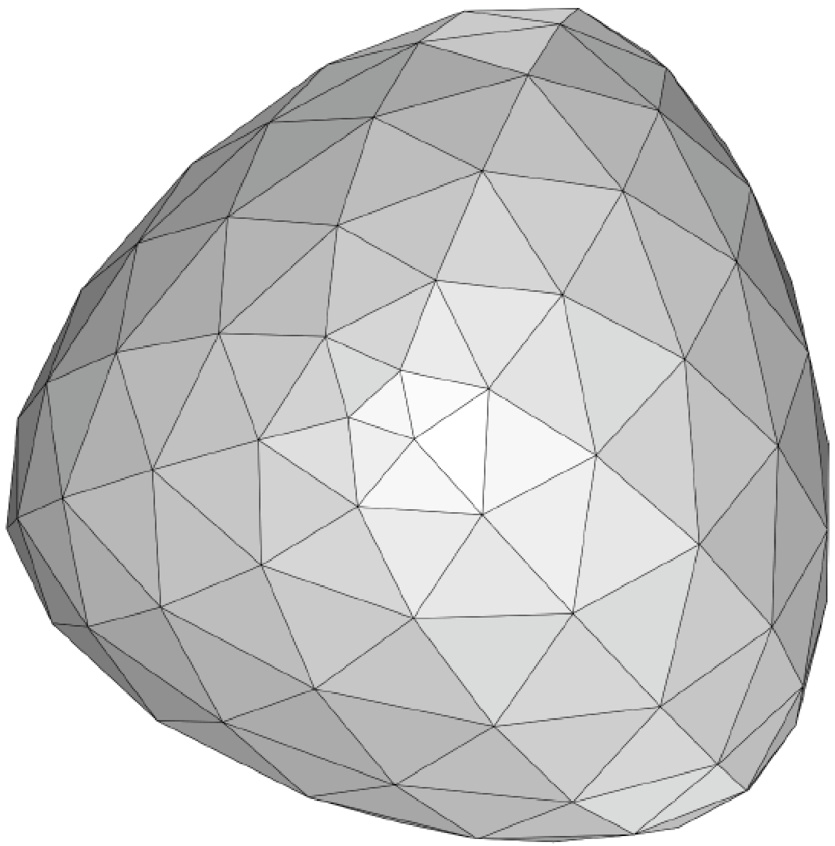
Results

Results



Mesh

Results



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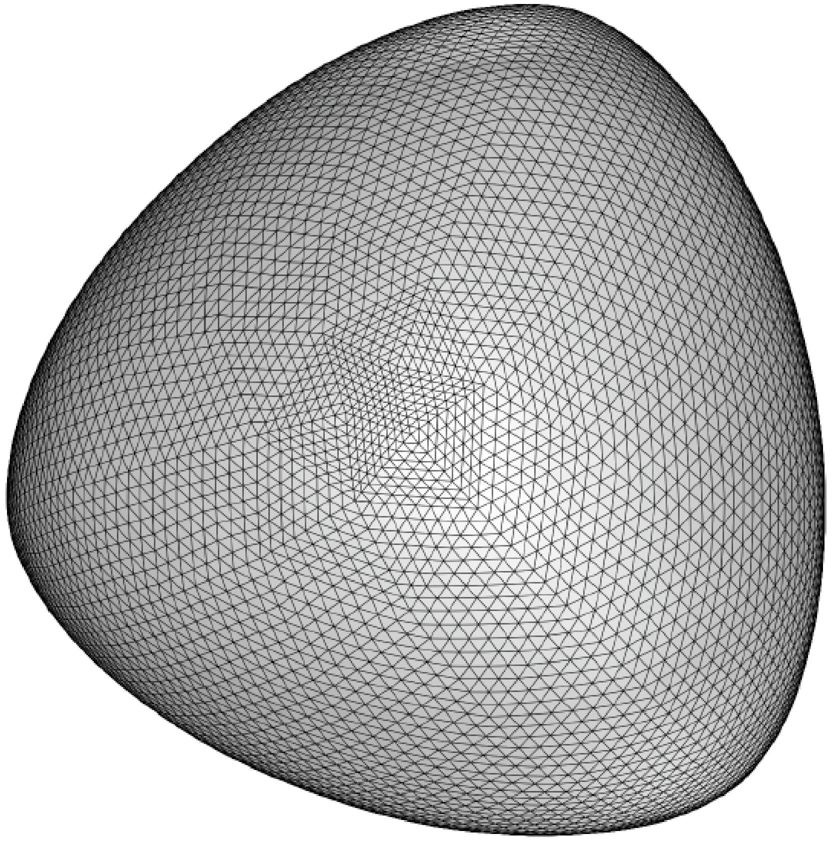

PN triangle

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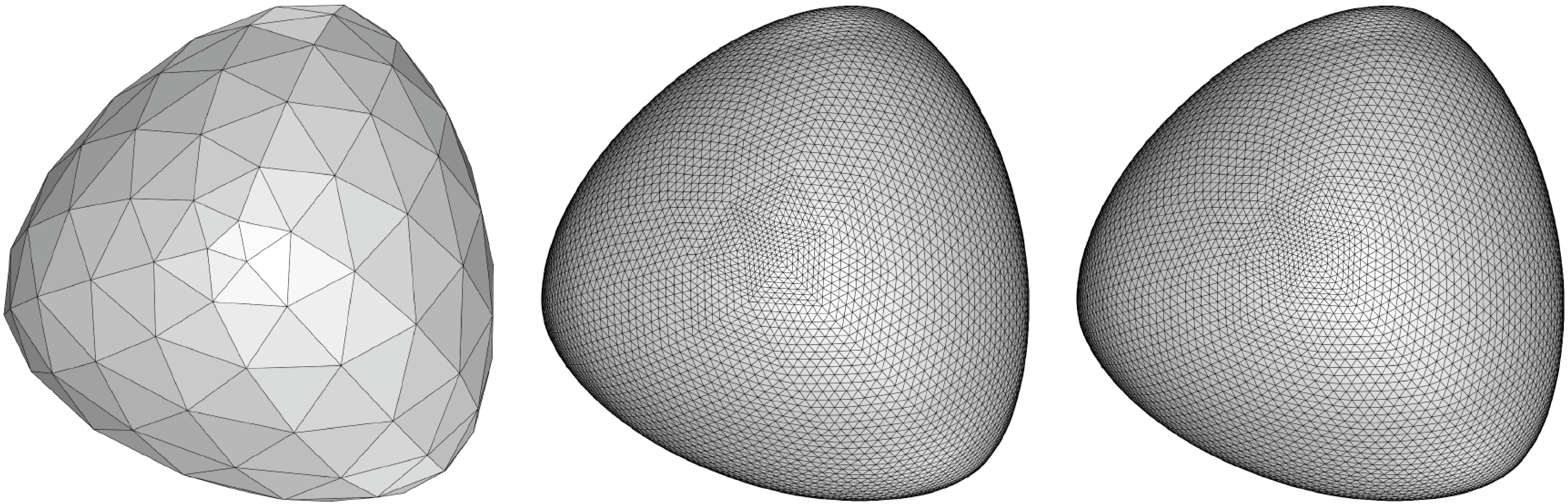

Mesh PN triangle PPS

Results

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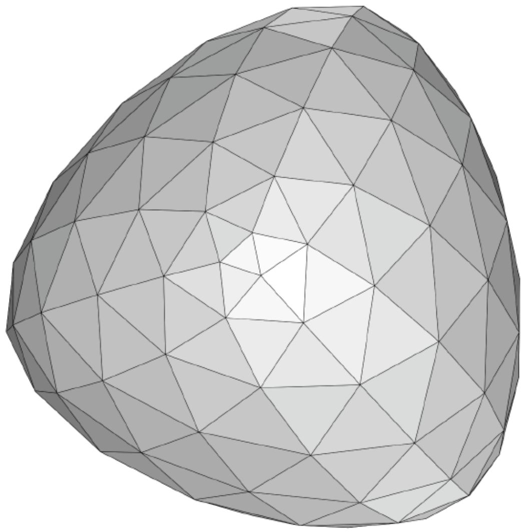

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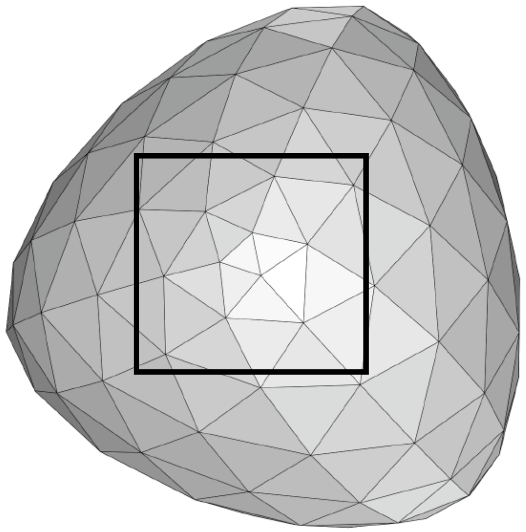

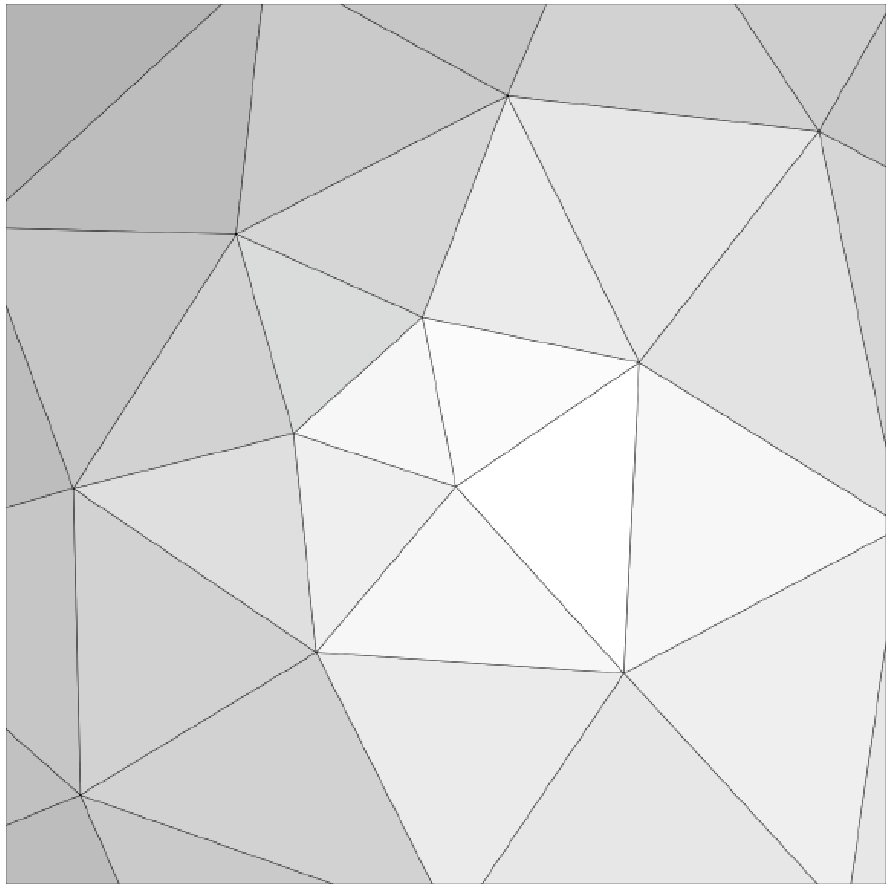

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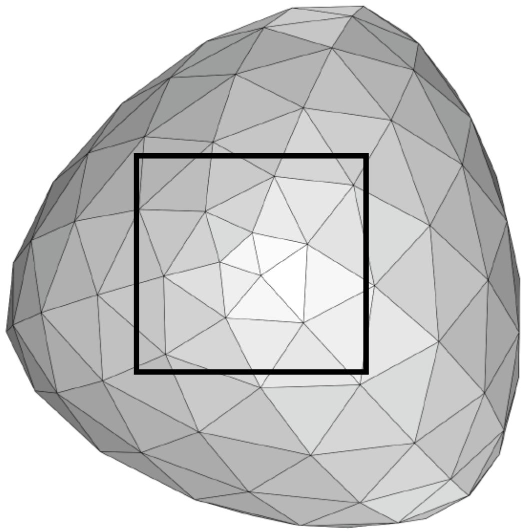

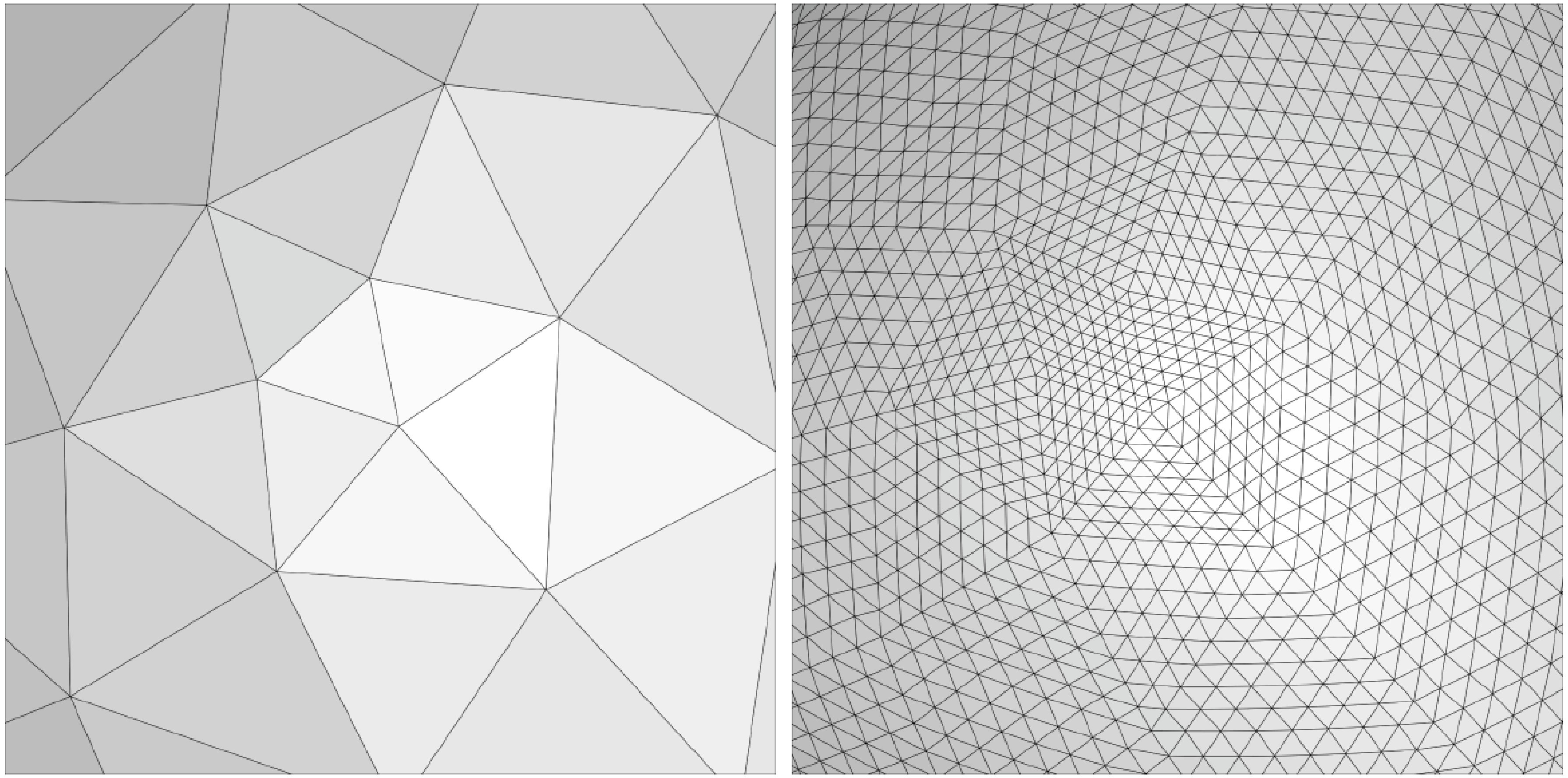

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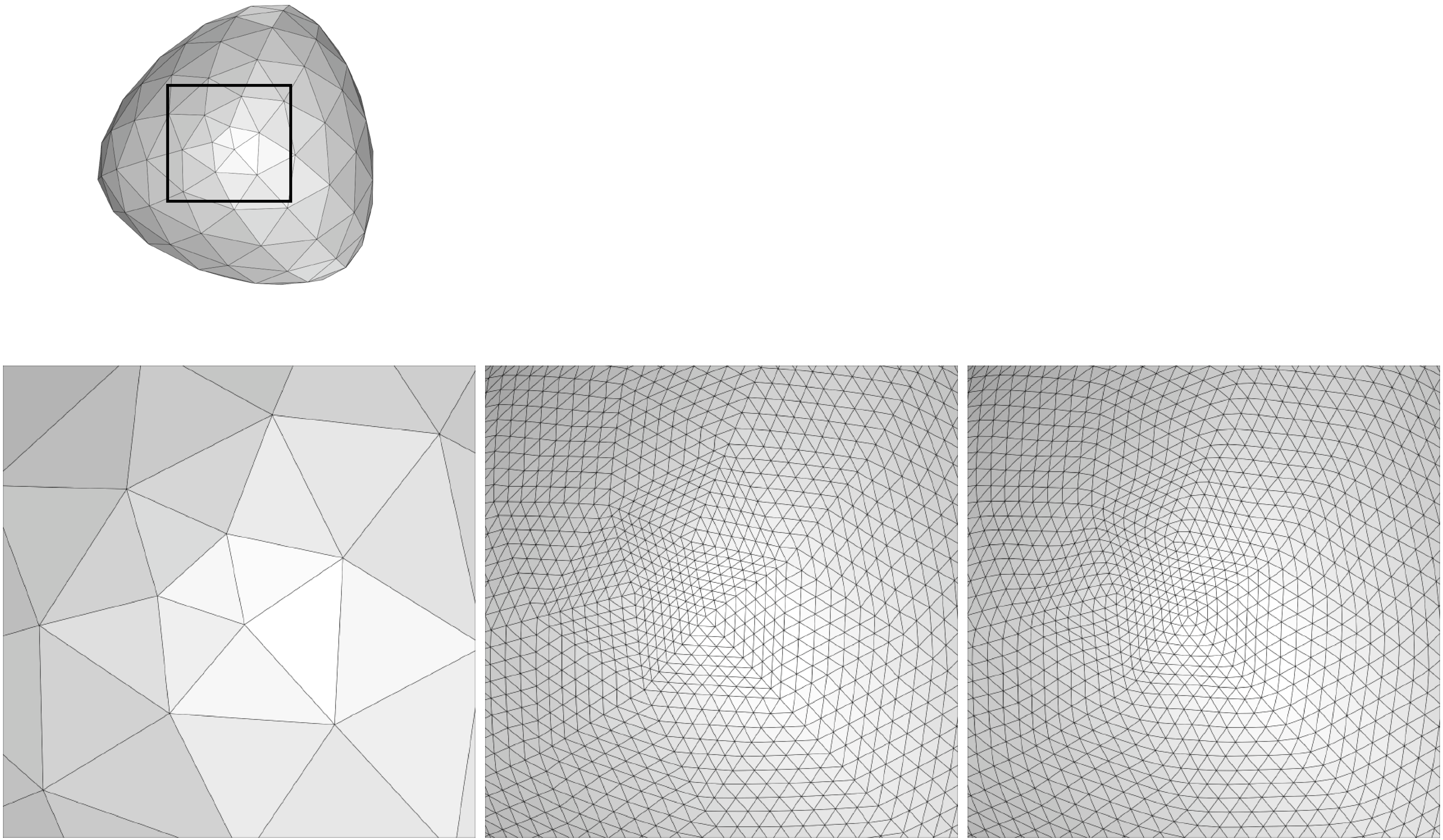

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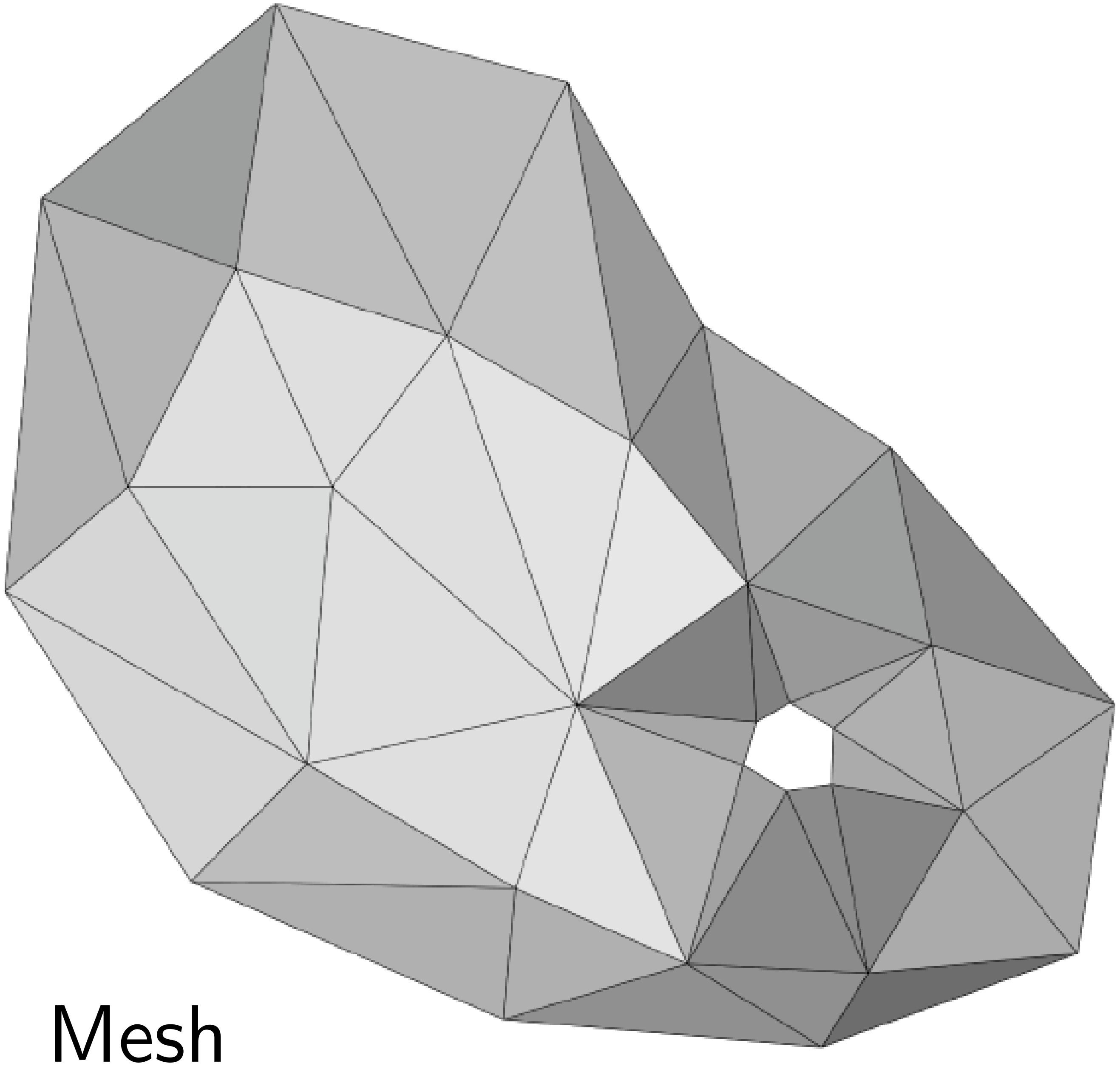

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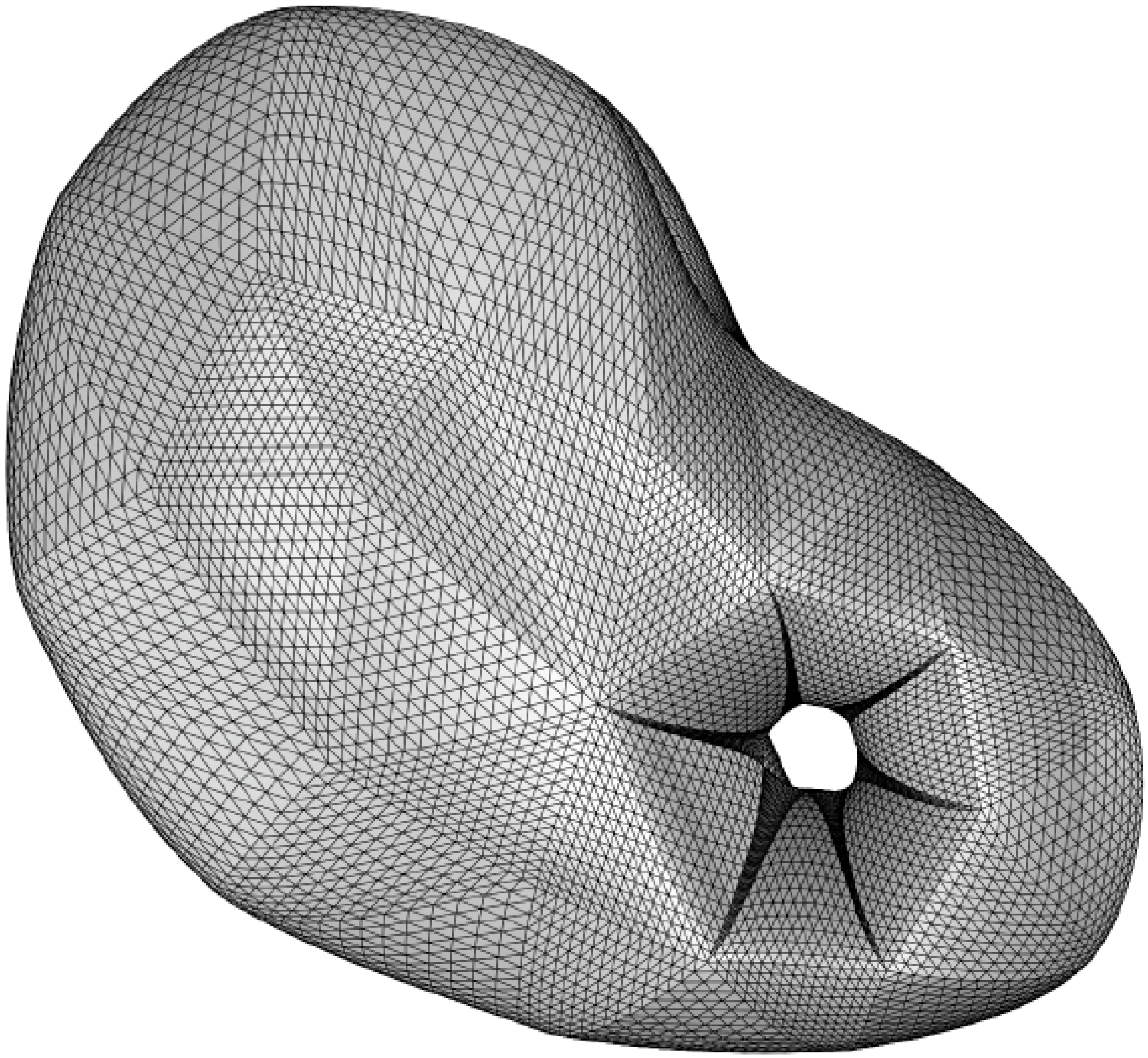

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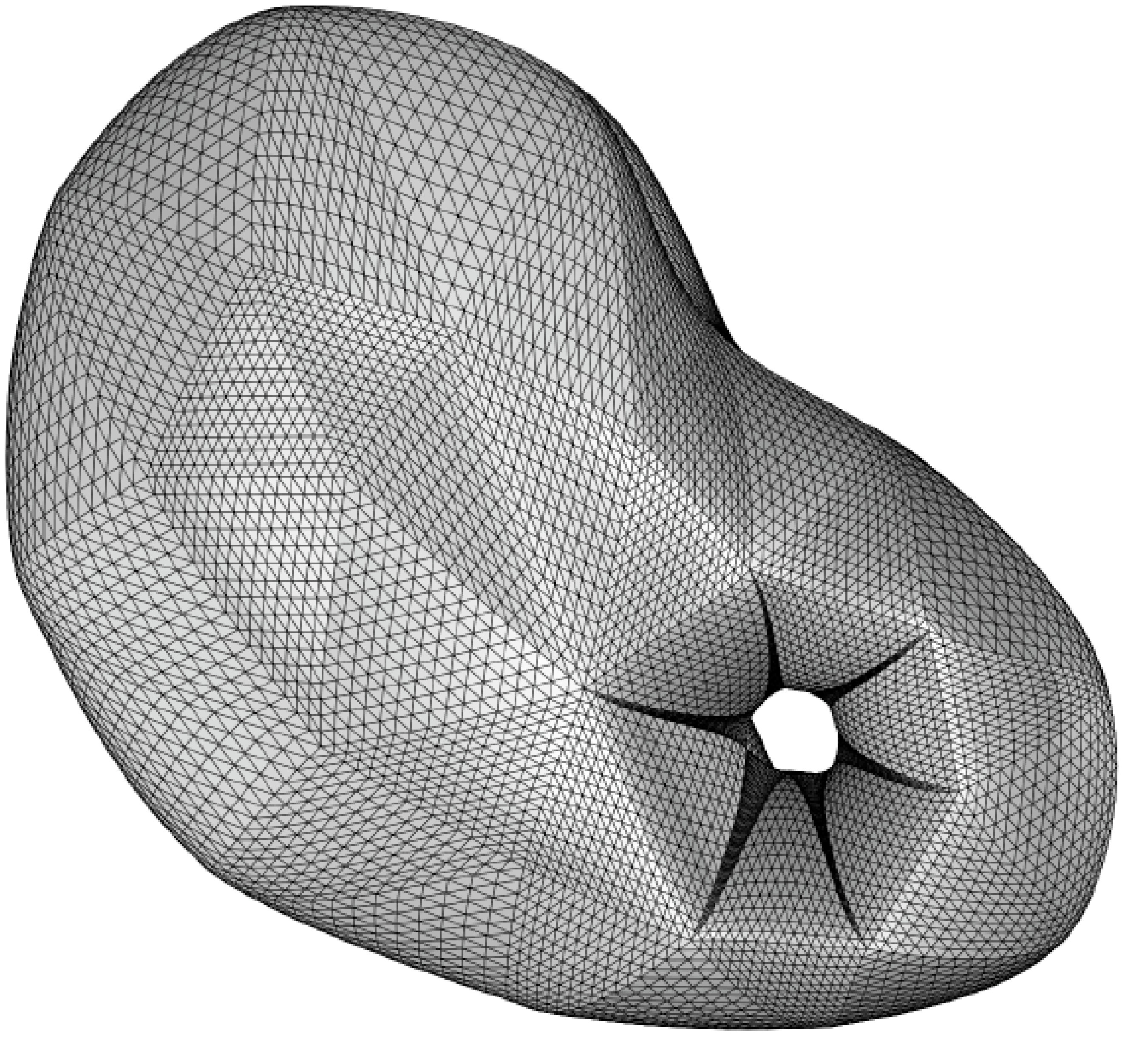
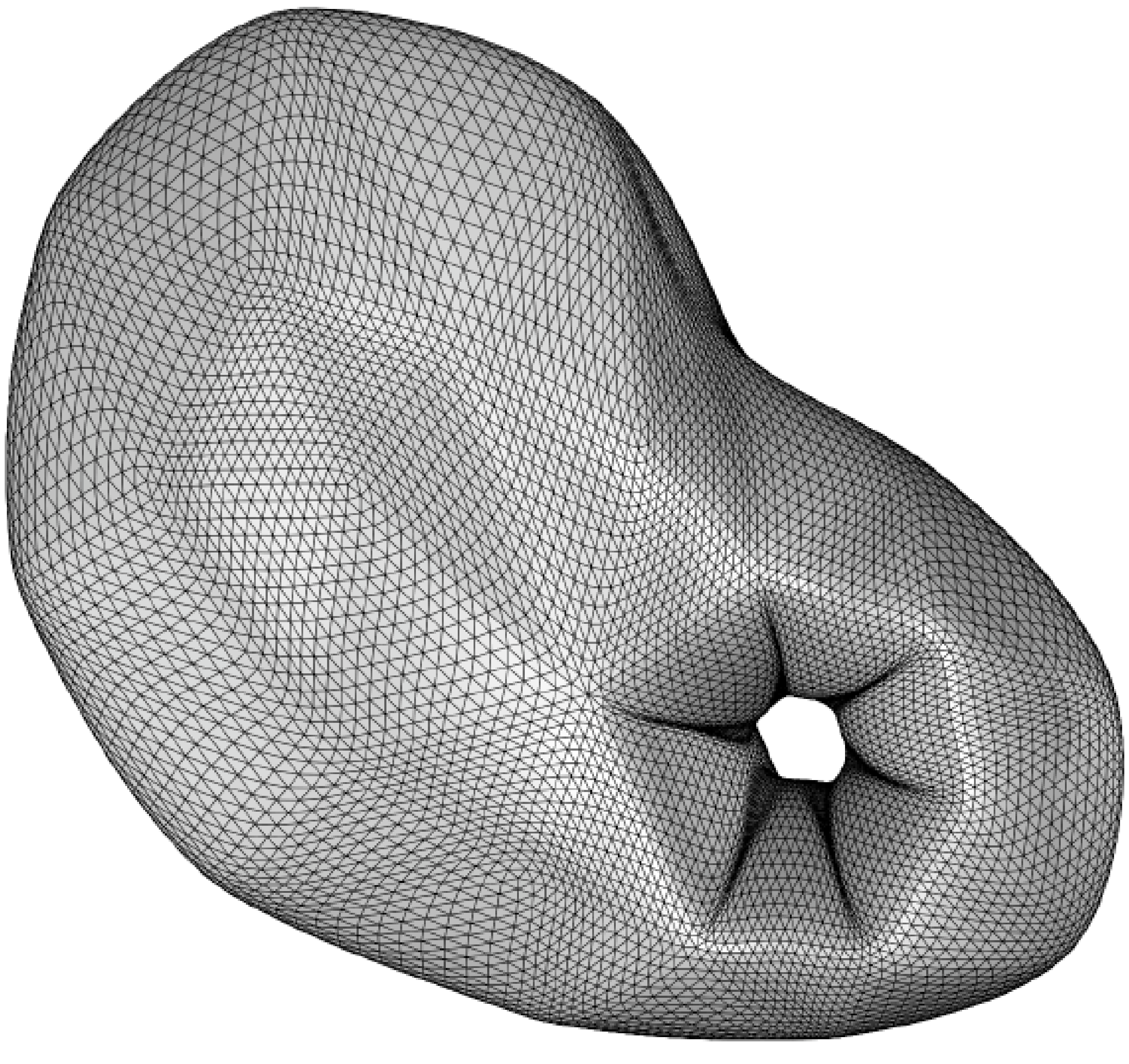


PPS



Results

Results

Mesh

Results

Mesh

PN triangle

Results

Mesh

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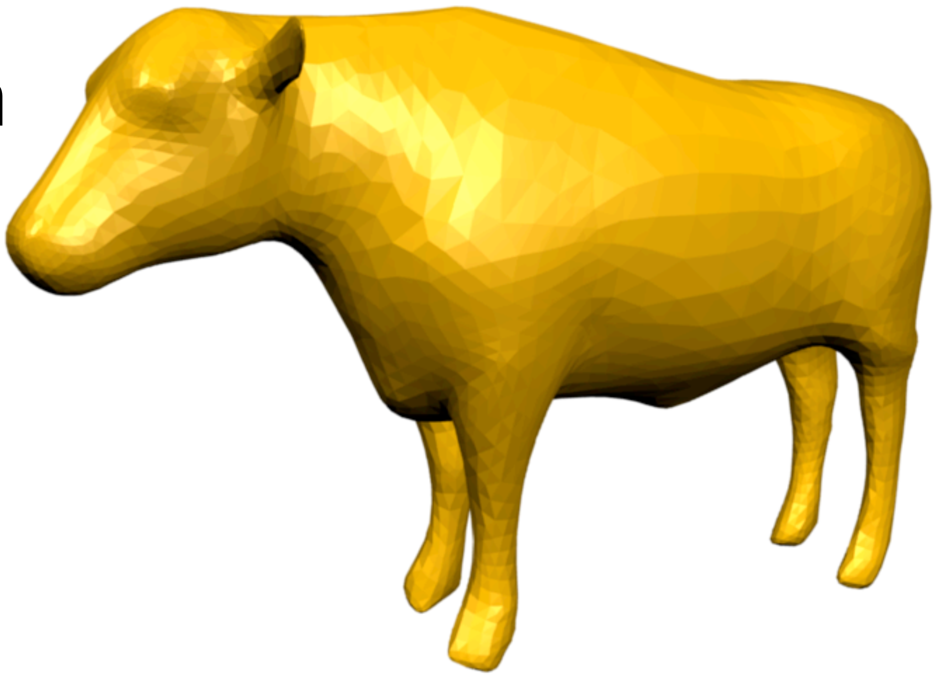

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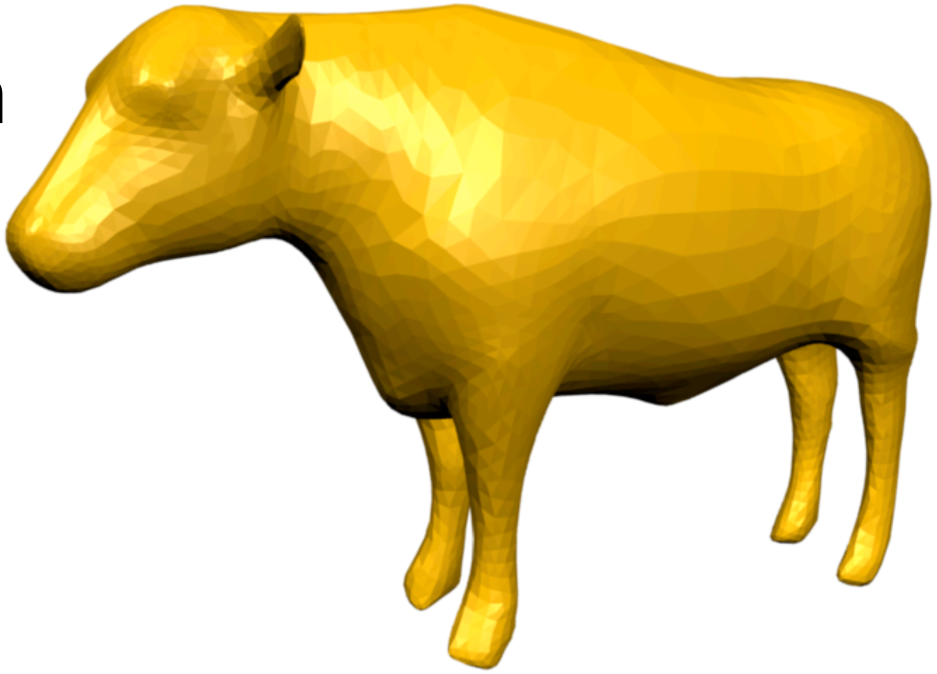

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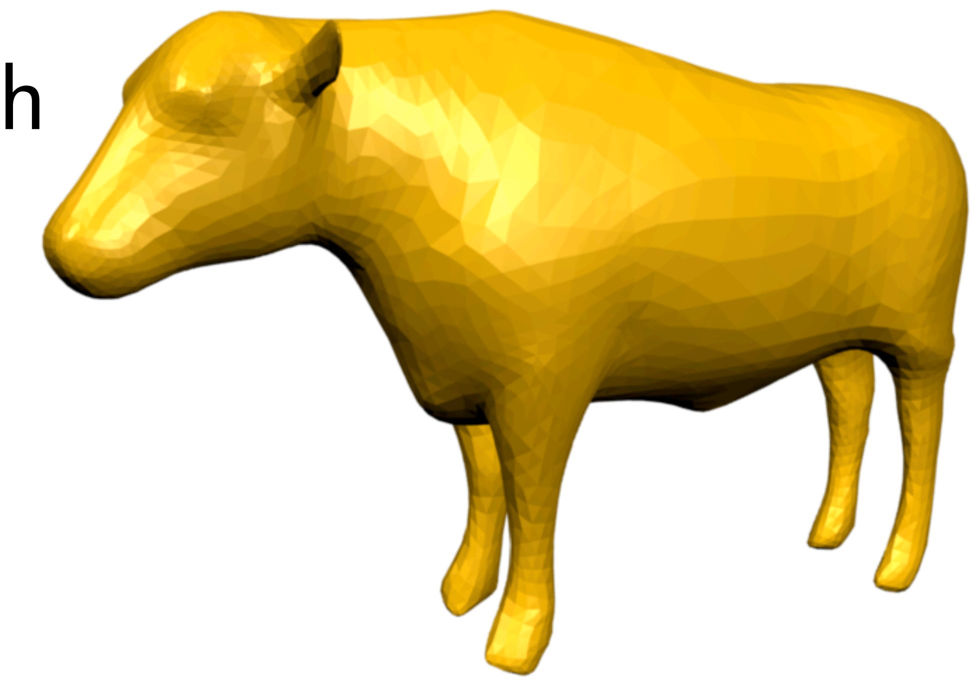

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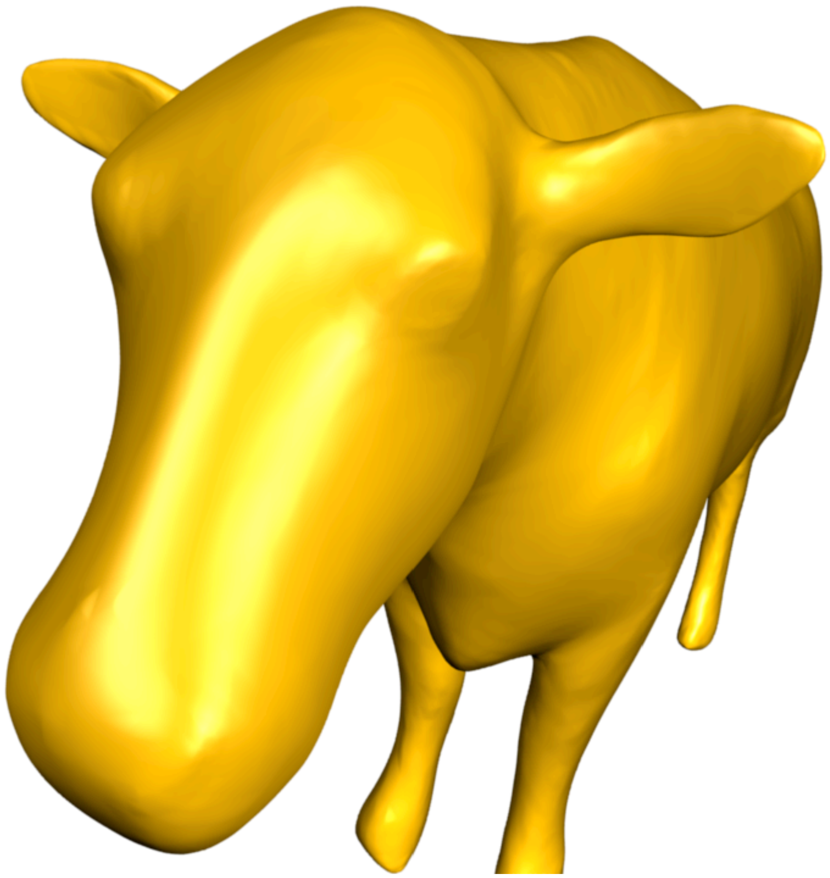

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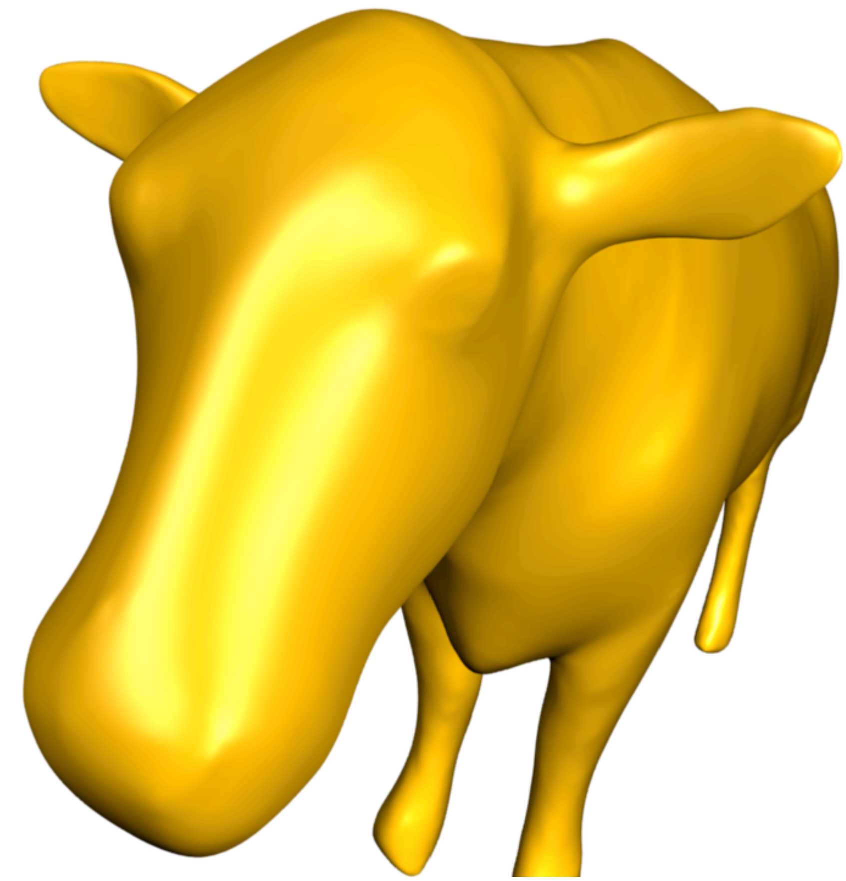

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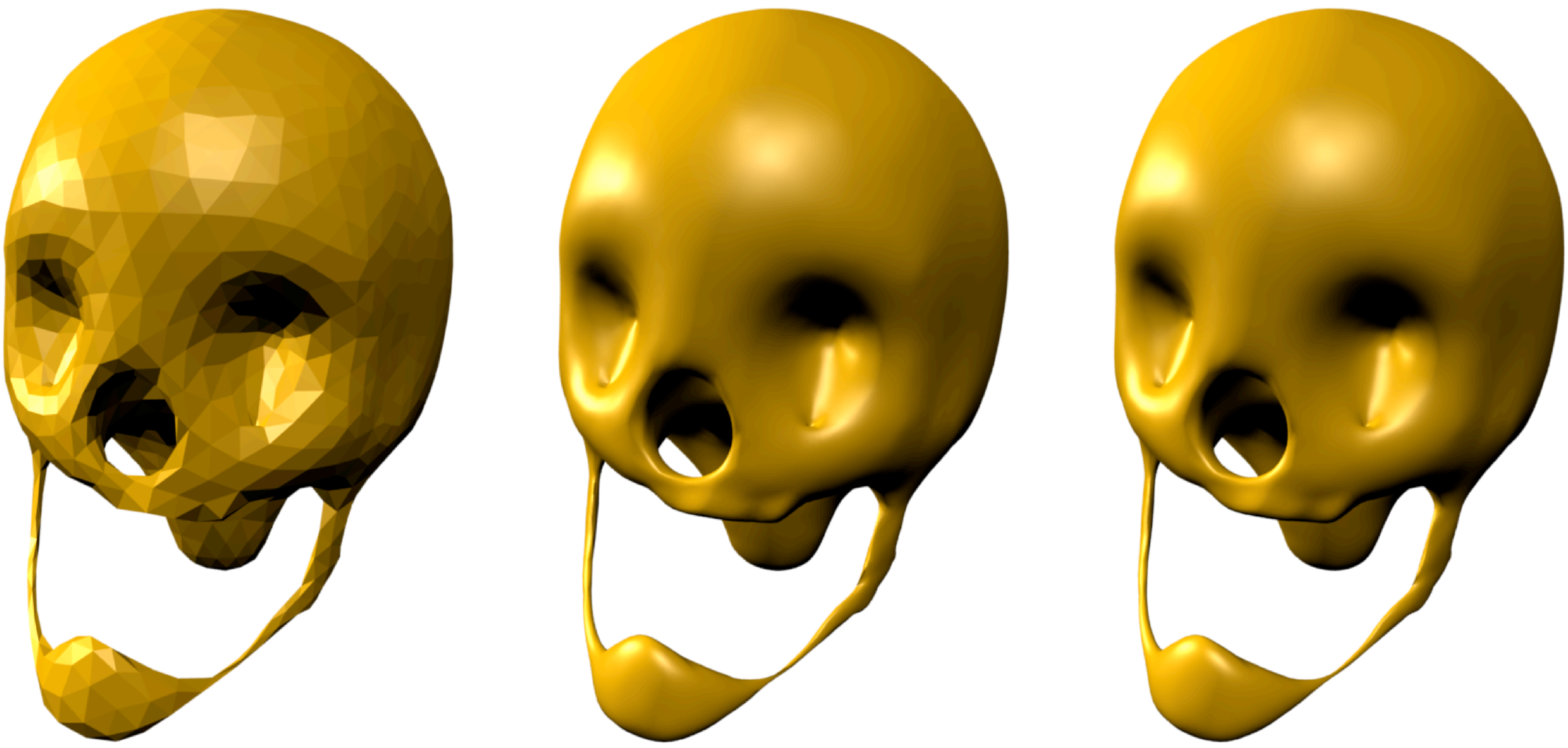

Mesh PN triangle PPS

Results

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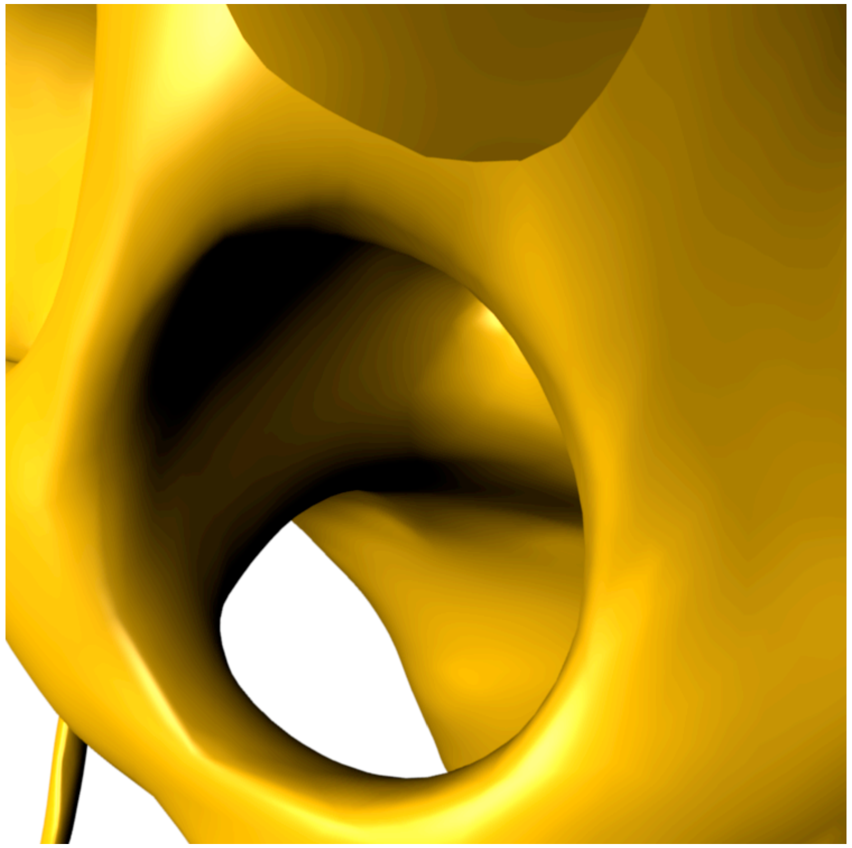

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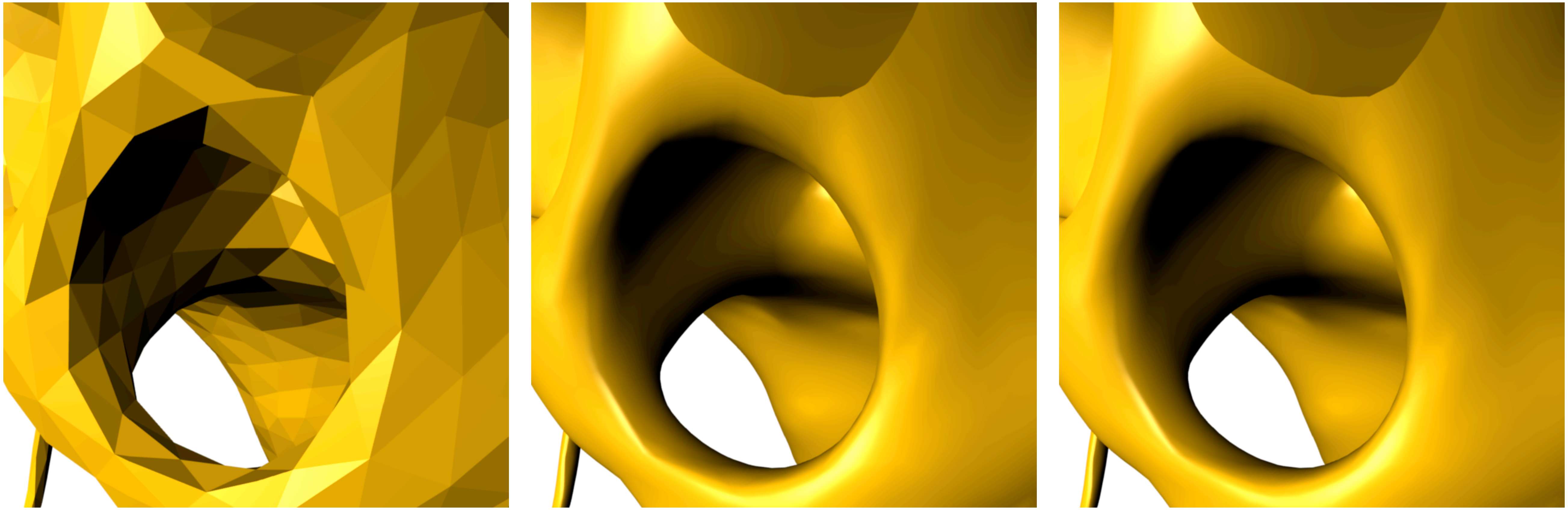

Mesh PN triangle PPS

Conclusions

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The image of our C^k parametric pseudo-surface is given by

$$M = \bigcup_{(\sigma,v)} \theta_{(\sigma,v)}(\Omega_{(\sigma,v)})\,.$$

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There are $3 \times n_t$ p-domains and Bézier patches in our construction, where n_t is the number of triangles of the input mesh, S_T.

Conclusions

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Unfortunately, the map $\theta_{(\sigma,v)}$ is NOT polynomial.

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OPEN PROBLEM: Can we make it polynomial?

Conclusions

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Recall that

$$\theta_{(\sigma,v)}(p) = \sum_{(\tau,u)\in J(p)} \nu_{(\tau,u)}(p) \cdot \psi_{(\tau,u)}(\varphi_{(\sigma,v)(\tau,u)}(p))\,,$$

where

$$\nu_{(\tau,u)}(p) = \frac{\gamma_{(\tau,u)}(\varphi_{(\tau,u)(\sigma,v)}(p))}{\sum_{(\eta,w)\in J(p)} \gamma_{(\eta,w)}(\varphi_{(\eta,w)(\sigma,v)}(p))}$$

and

$$J(p) = \{(\eta, w) \in I \mid p \in \Omega_{(\sigma,v)(\eta,w)}\}\,.$$

Conclusions

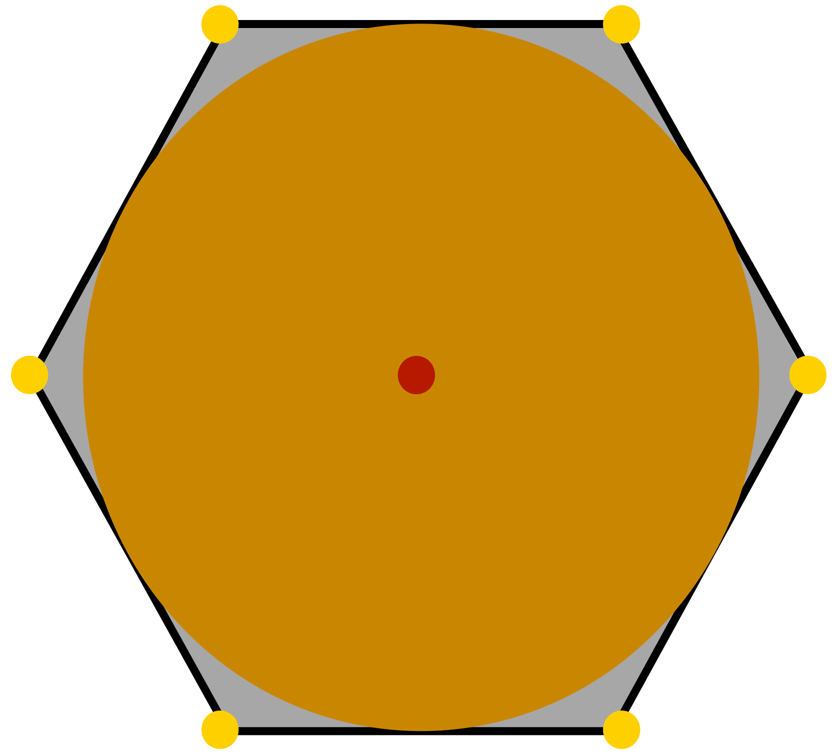
We can easily make $\gamma_{(\tau,u)}$ a C^k rational polynomial, for any finite k.

However, the difficult lies in making $\varphi_{(\tau,u)(\sigma,v)}$ (rational) polynomial!.

Conclusions

Conclusions

We can create a much simpler construction by letting the p-domains be the inscribed circles of the P-polygons, as shown below:



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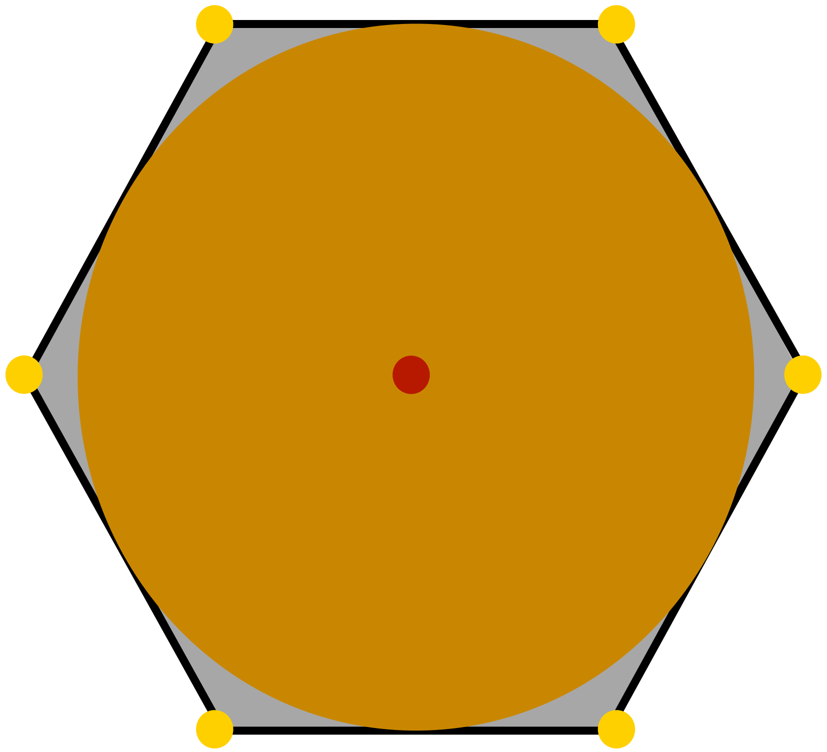
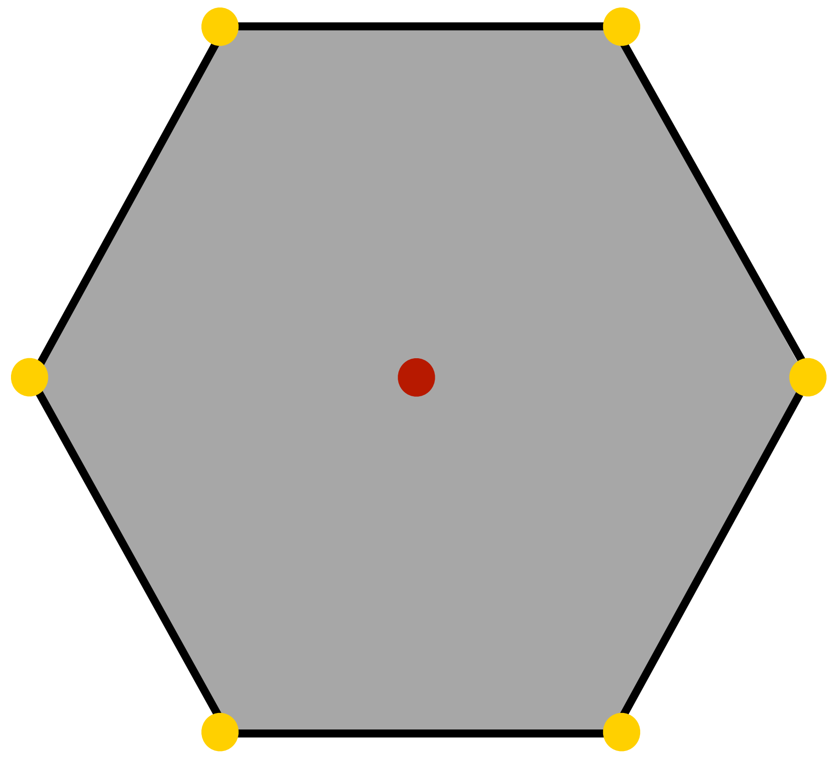


The transition maps do not change, but the shape functions do!

Conclusions

Conclusions

Why didn't we let the interior of the P-polygons be the p-domains?



Conclusions

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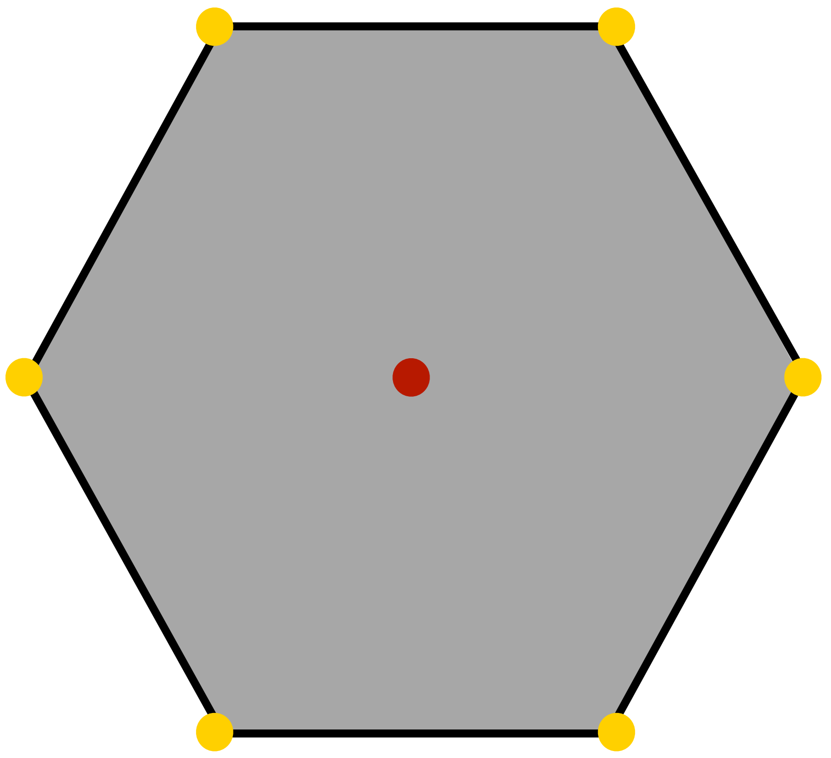
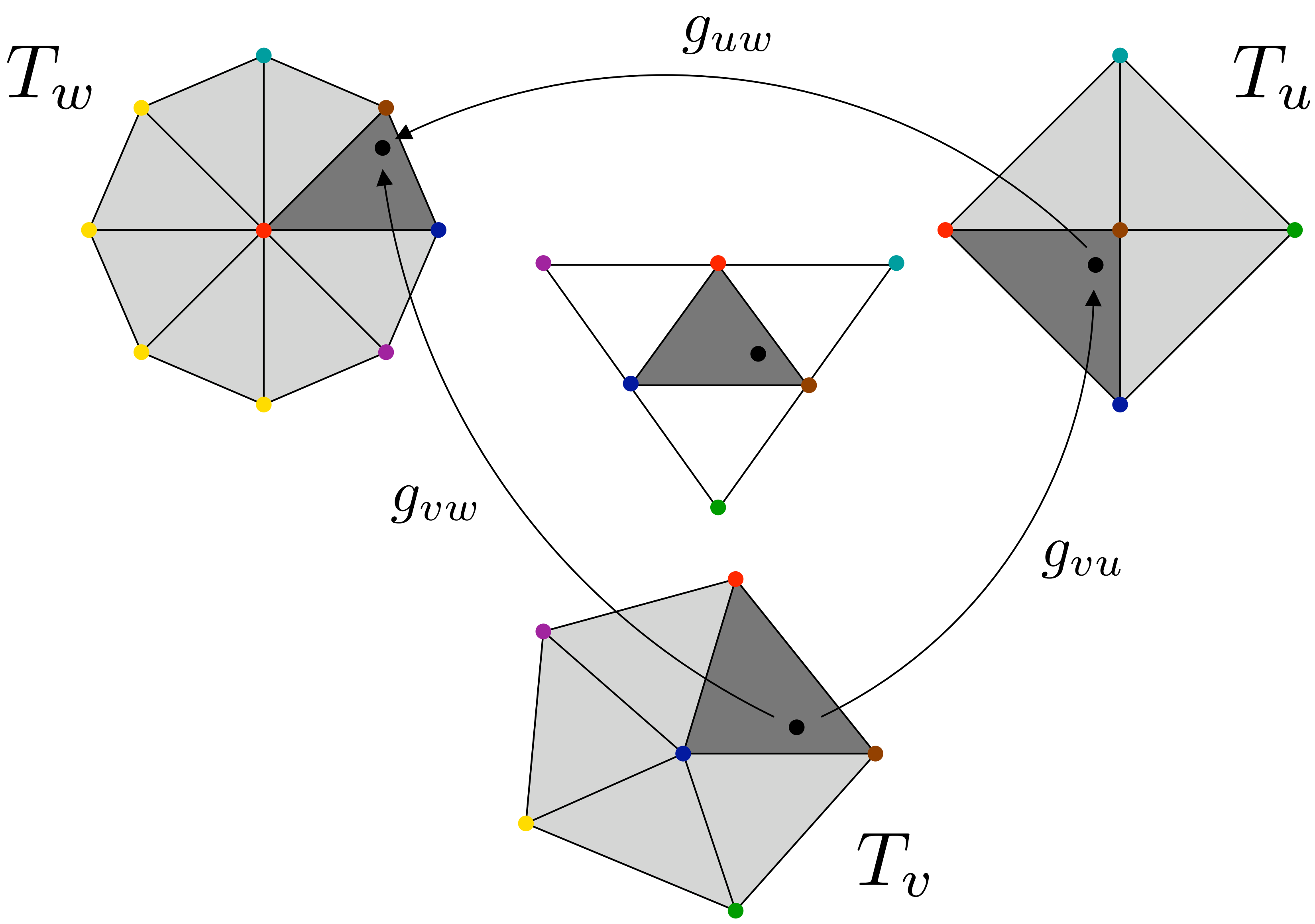


Simple answer: we failed to figure out the transition maps!

Conclusions

Conclusions

OPEN PROBLEM: Can you find a **simple** C^∞ bijective map g satisfying $g_{vw} = g_{uw} \circ g_{vu}$ (this has to do with the cocycle condition)?



Conclusions

Conclusions

For a good survey on the existing constructions, see

- Cindy M. Grimm and Denis Zorin. Surface Modeling and Parametrization with Manifolds. In ACM SIGGRAPH 2006 Courses (SIGGRAPH'06), pages 1-81, New York, NY, USA, 2006. ACM Press.

Adaptive Manifold Fitting (Part I)

Luiz Velho
IMPA

Outline

- Fitting Surfaces to Very Large Meshes
- Multiresolution Operators
- Building Base Meshes by Simplification
- Adaptive Mesh Refinement
- Conclusions

Surface Fitting

- Very Large Meshes (10^6 vertices)
  - Challenging Problem!

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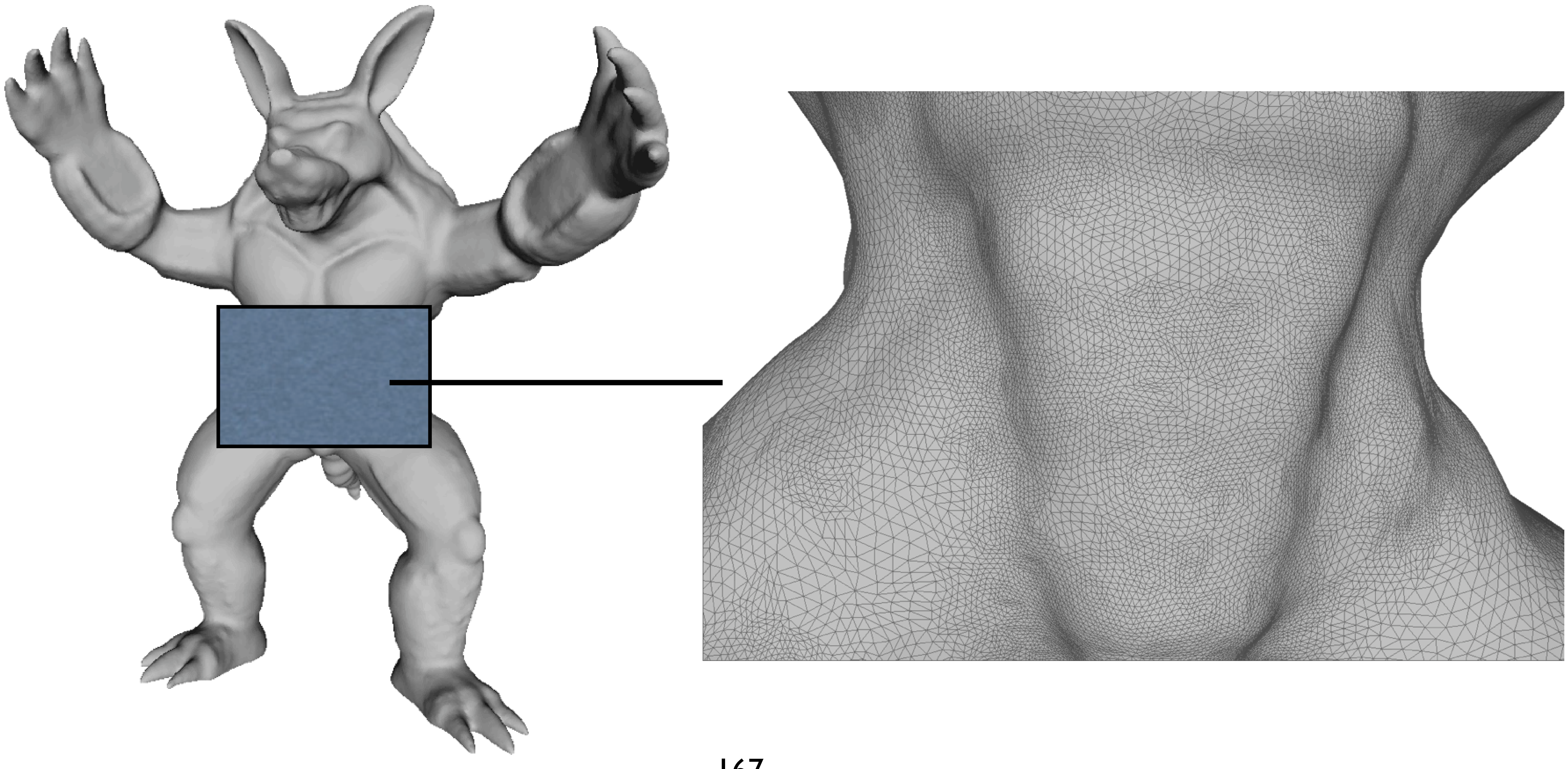


Manifolds and Fitting

- Basic Setting
  - Gluing Data proportional to Mesh Size

- Problem: *Very Large Meshes*
  - Computationally Inefficient
  - Do not Exploit Approximation Power
- Solution:
  - Adaptation

Adaptive Fitting

- Optimization Formulation:
  - Given an Approximation Error ϵ
  - Find $\mathcal{M}$ with Smallest Number of Charts

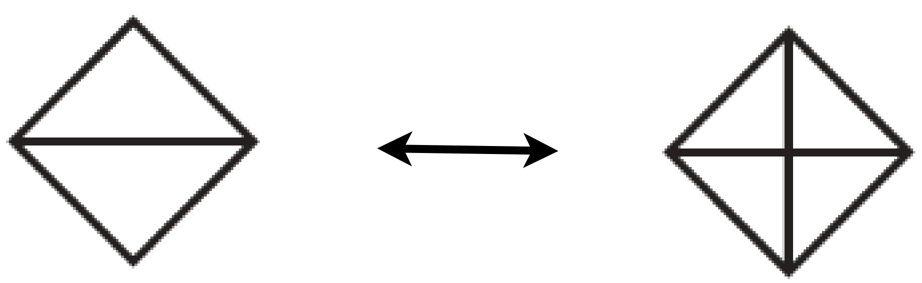
- Strategy:
  - Combine
    - Multiresolution Structure
    - Manifold Surface Approximation

Multiresolution Framework

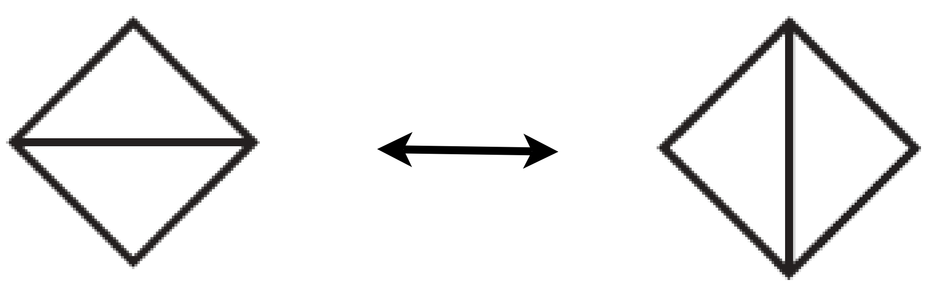
- Simplicial Multi-triangulation
  - Stellar Theory
- Building Base Meshes
  - Surface Simplification
- Adaptive Fitting
  - *4-8* Refinement

Stellar Theory

- Topological Operators
- Edge Split and Weld
  - Change Mesh Resolution



- Edge Flip
  - Change Mesh Connectivity

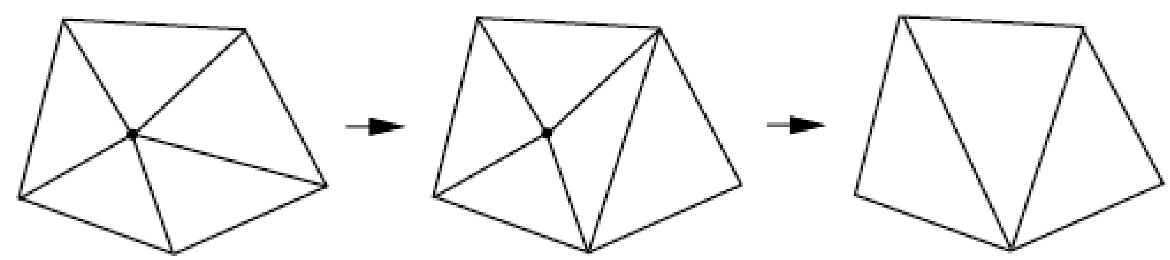


Stellar Simplification

- Basic Elements:
  I. Operator Factorization

     - Edge Collapse

     - Flip + Weld



  II. Quadric Error Metric

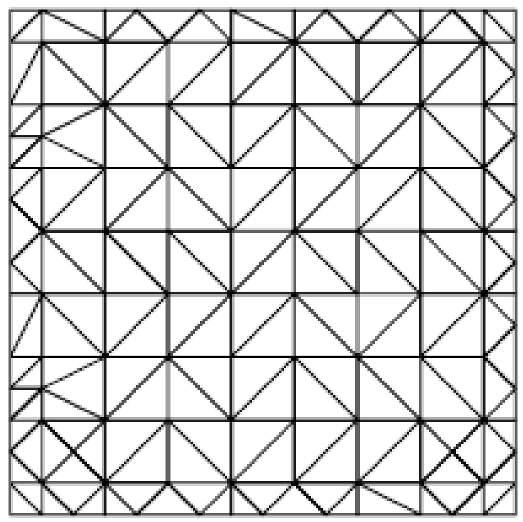
Basic Algorithm

- Repeat for *N* Resolution Levels
  1. Rank Vertices Based on Quadric Error
  2. Select Independent Set of Clusters
  3. Simplify Mesh using Stellar Operators

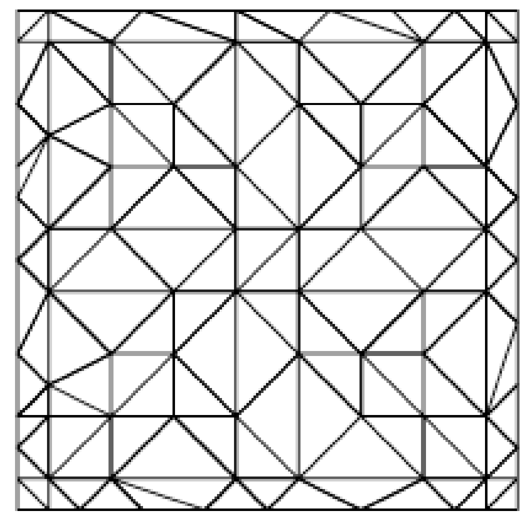
✳ *Properties*

- *Logarithmic Height*
- *Good Aspect Ratios*

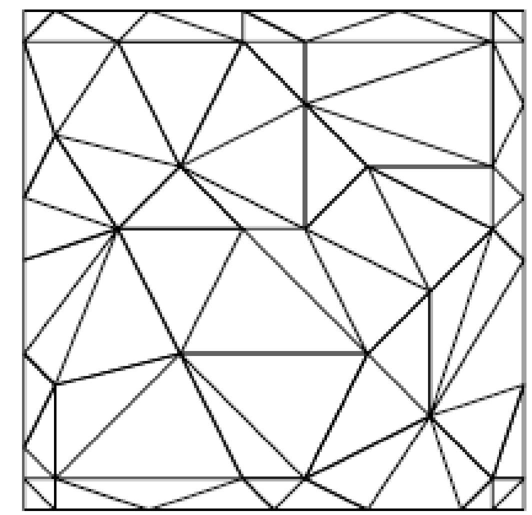
Example 1: Plane



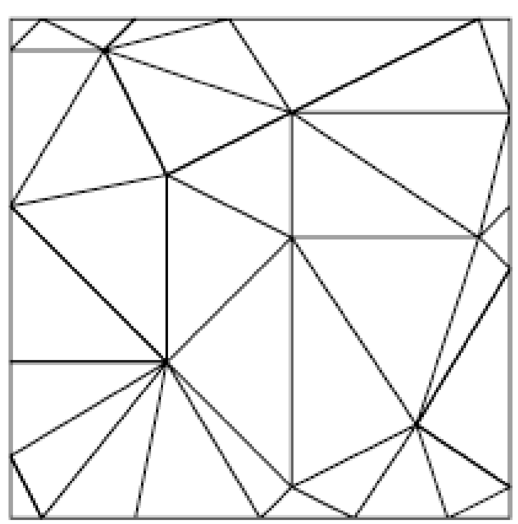
(a) original mesh



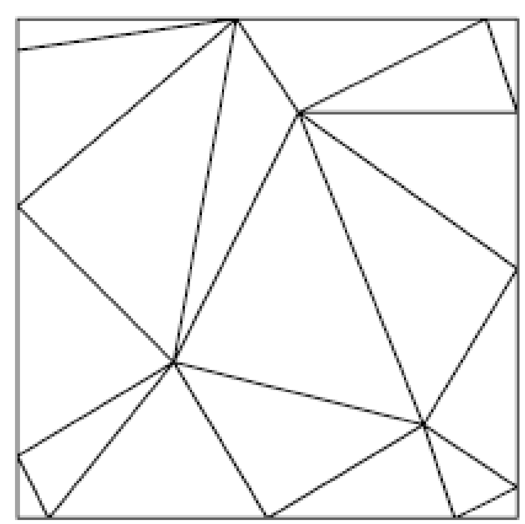
(b) level 1



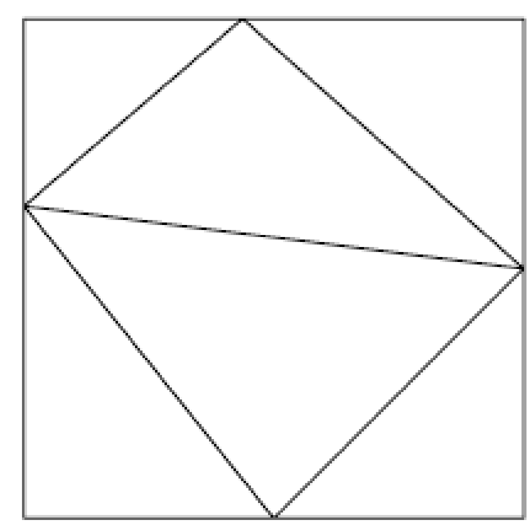
(c) level 3



(d) level 5

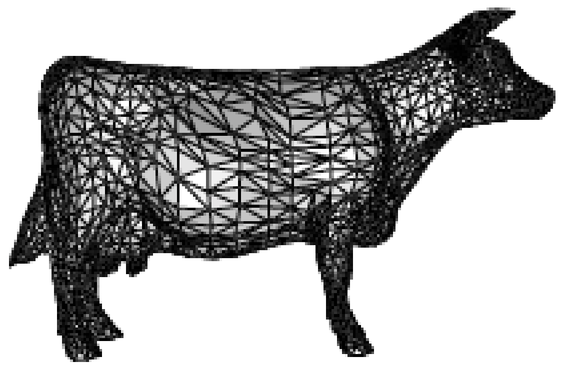


(e) level 7

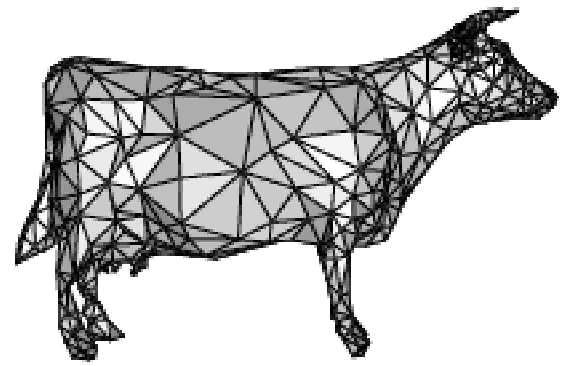


(f) level 9

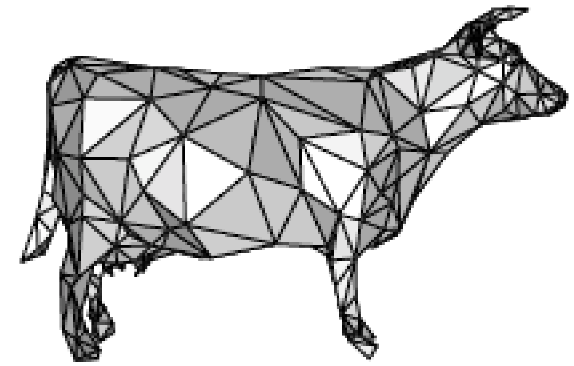
Example 2: Cow



(a) original mesh



(b) level 1



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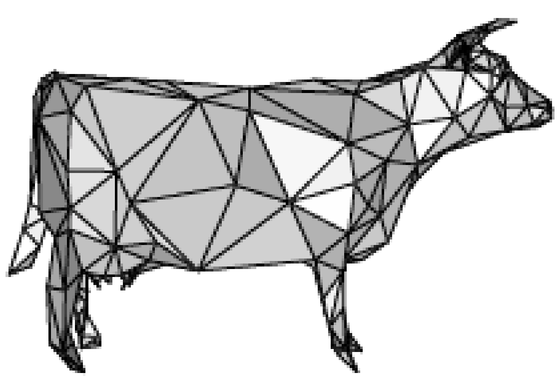
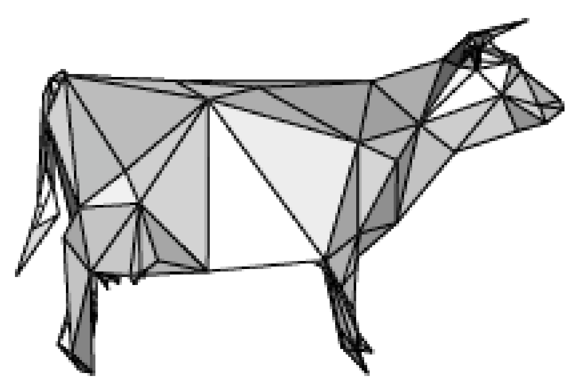


(d) level 5



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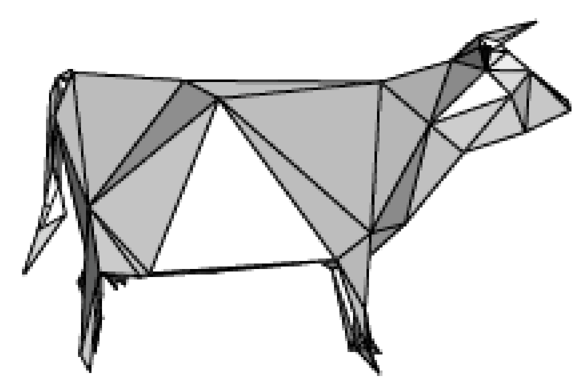
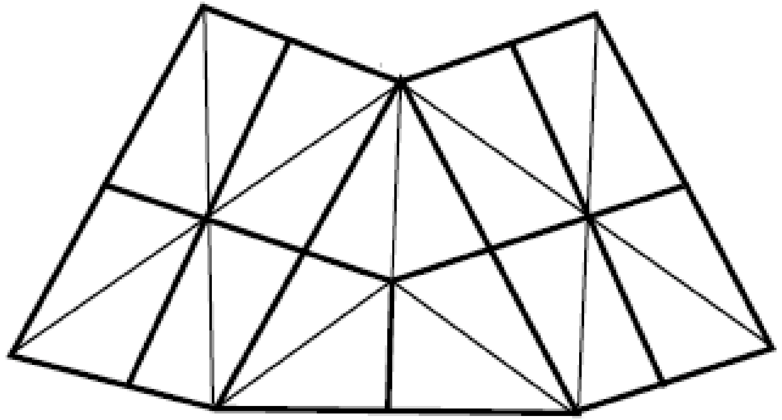


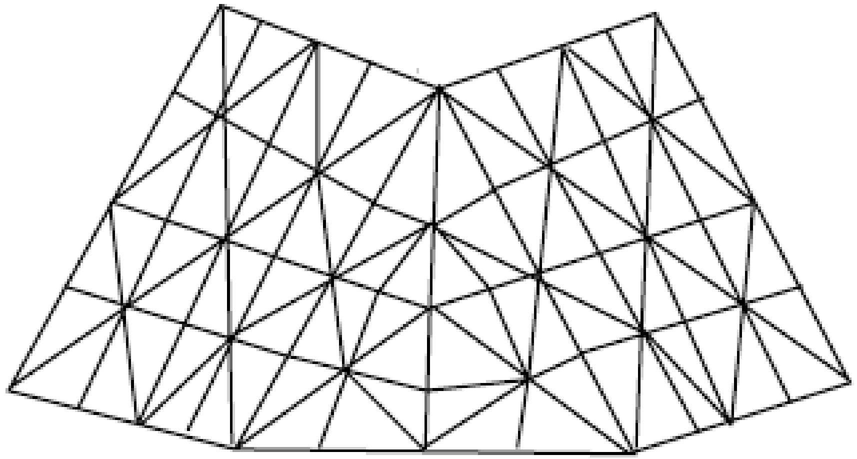
(f) level 9

Variable Resolution Mesh

- Underlying Semi-Regular Structure
  - Tri-quad Base Mesh

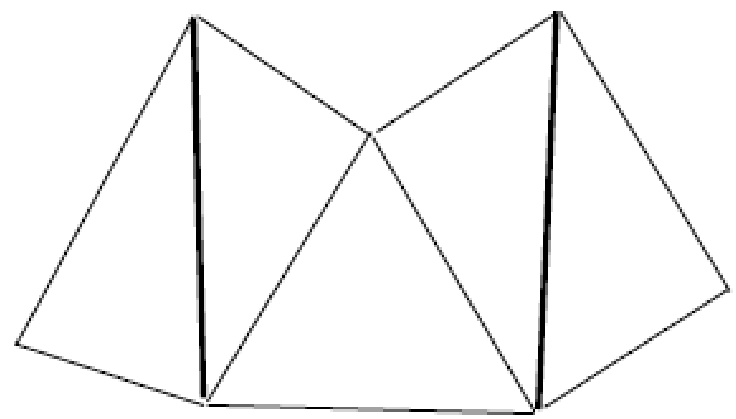


  - *4-8* Subdivision

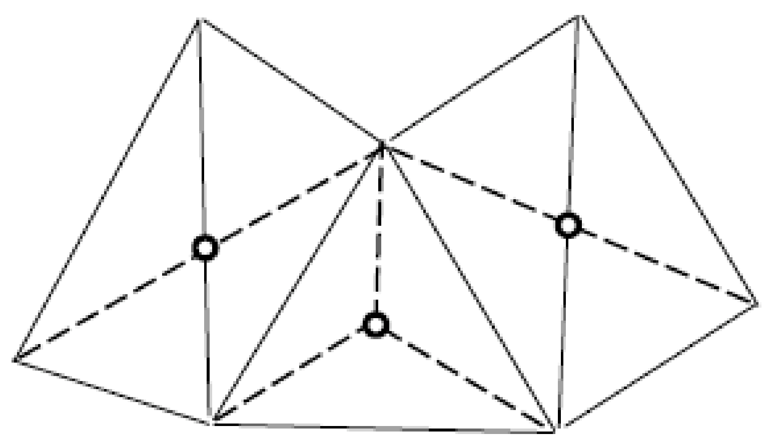


Building the Base Mesh

1. Two-Face Clusters + Single Triangles

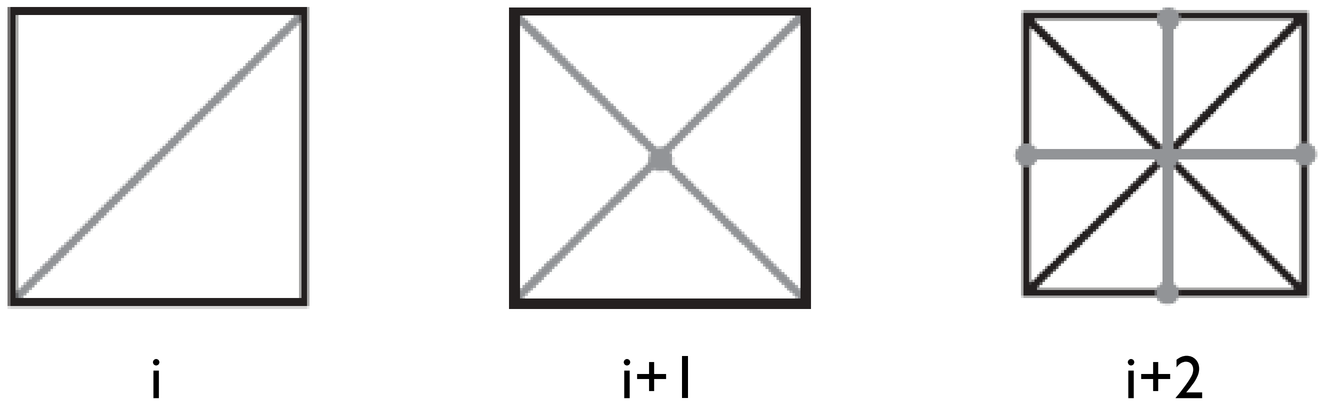


2. Barycenter Subdivision

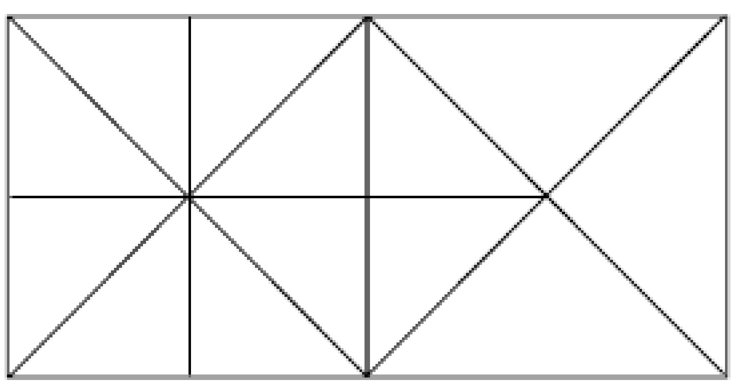


4-8 Subdivision

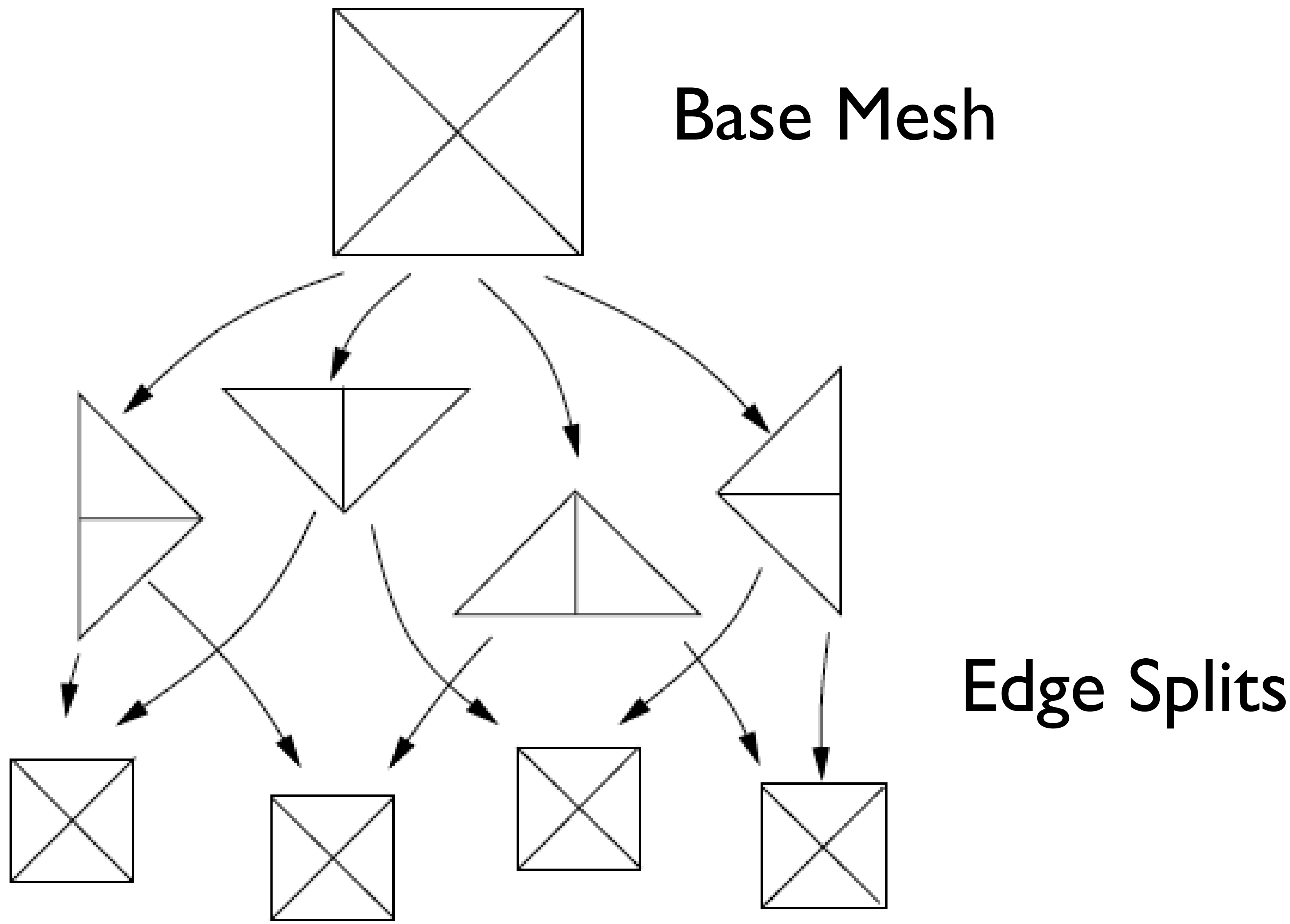
- Interleaved Binary Subdivision



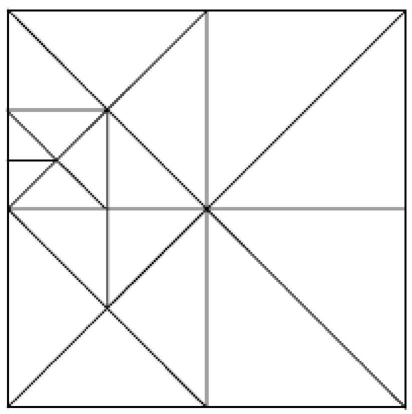
- Non-Uniform Refinement

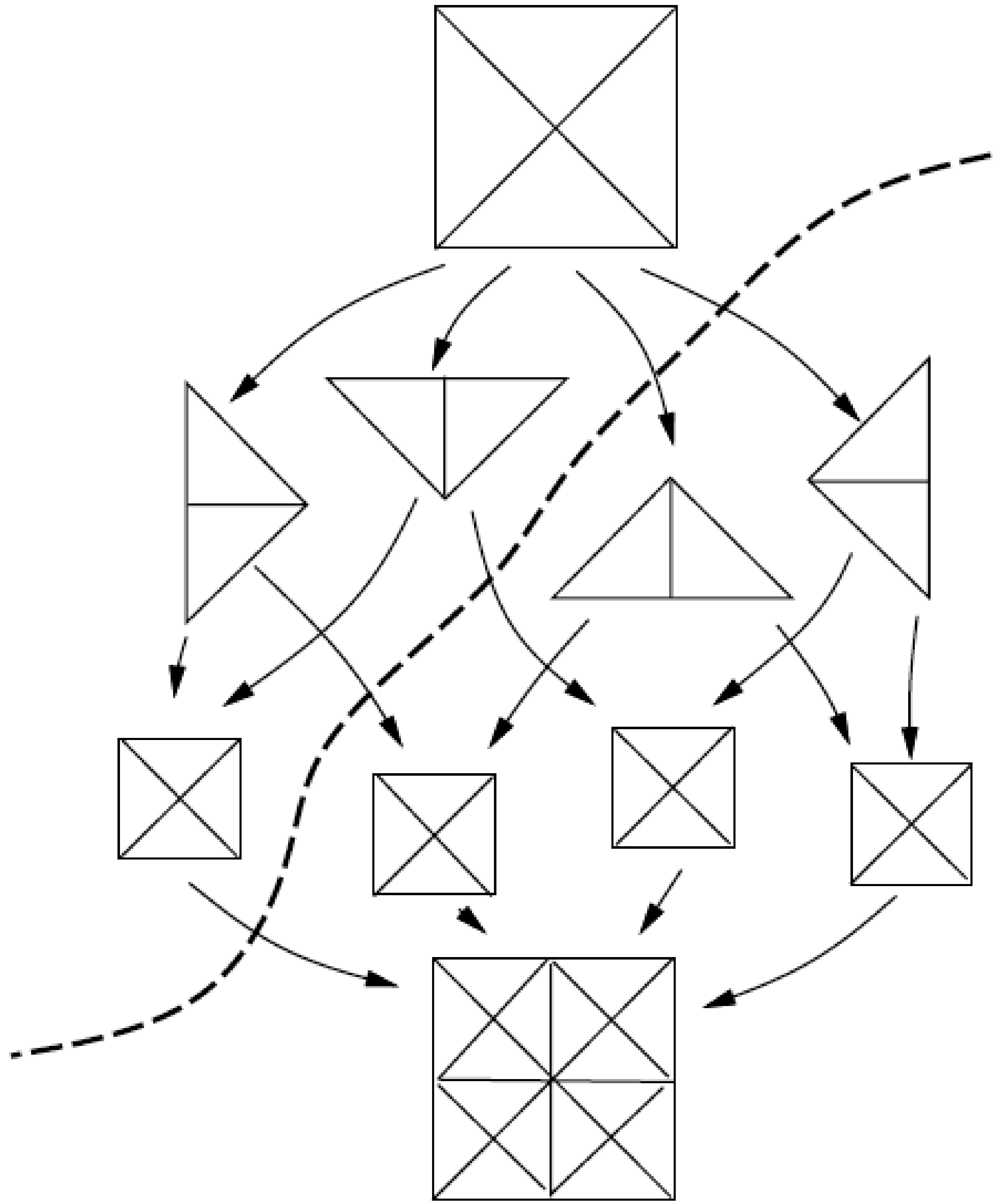


Binary Multi-Triangulation

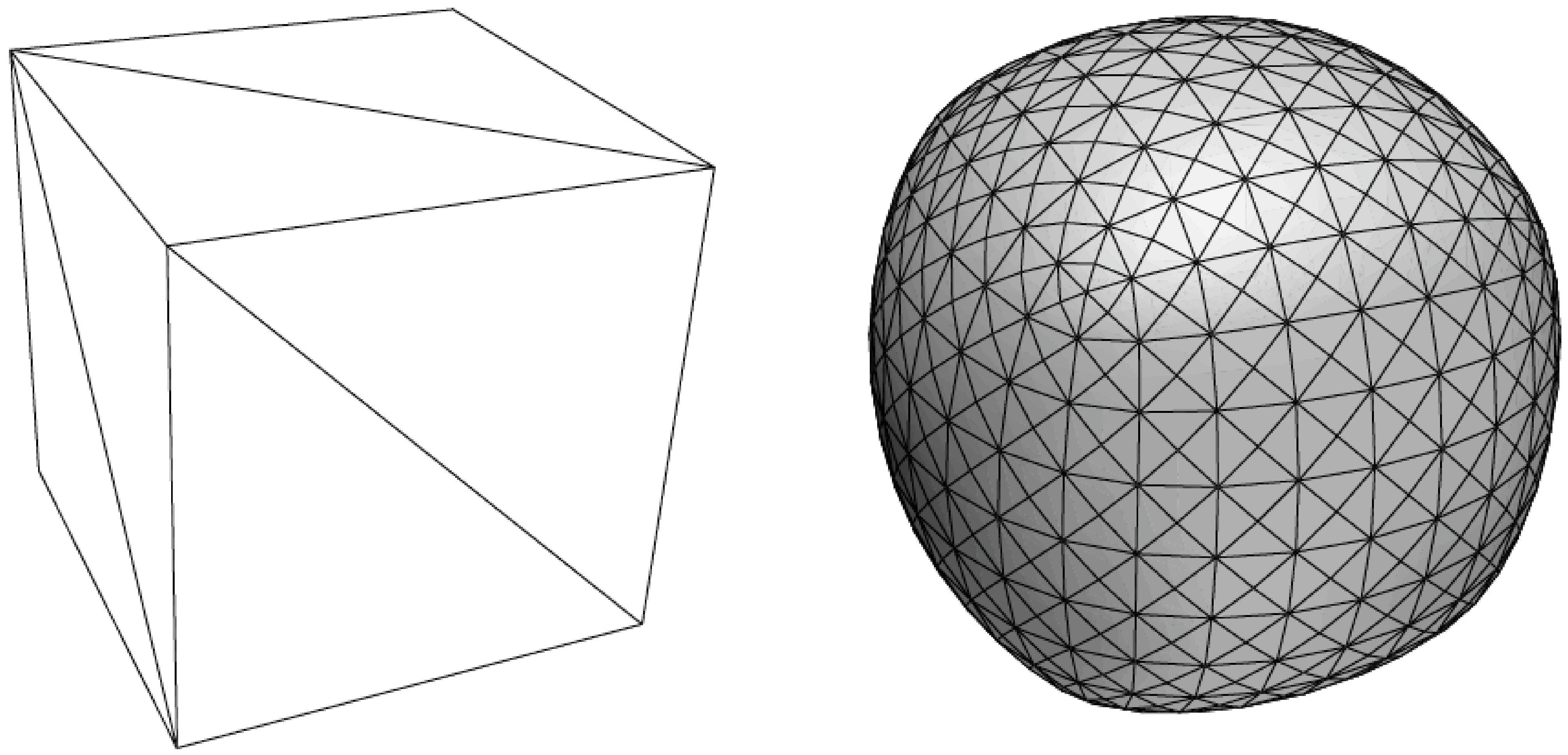


Adaptive Refinement



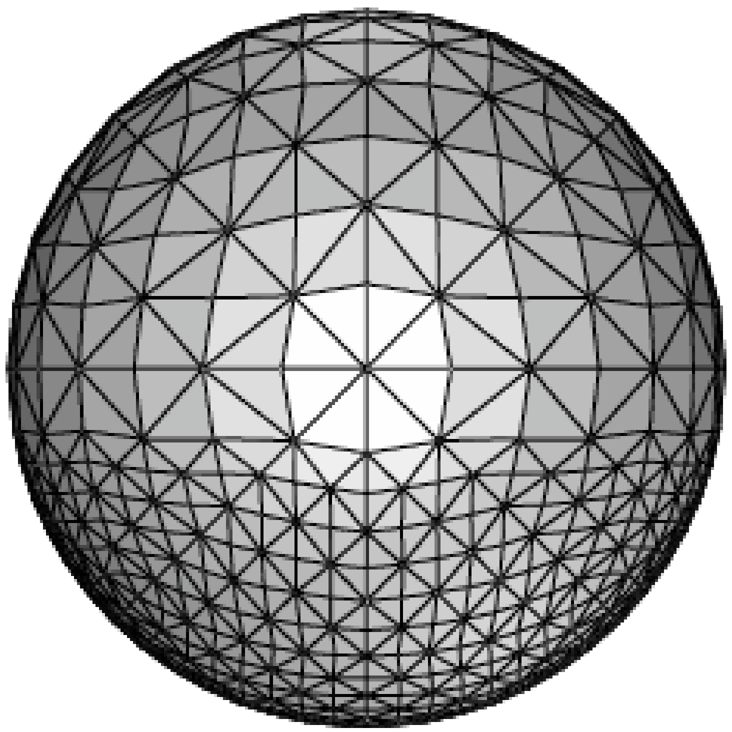


Example I: Uniform

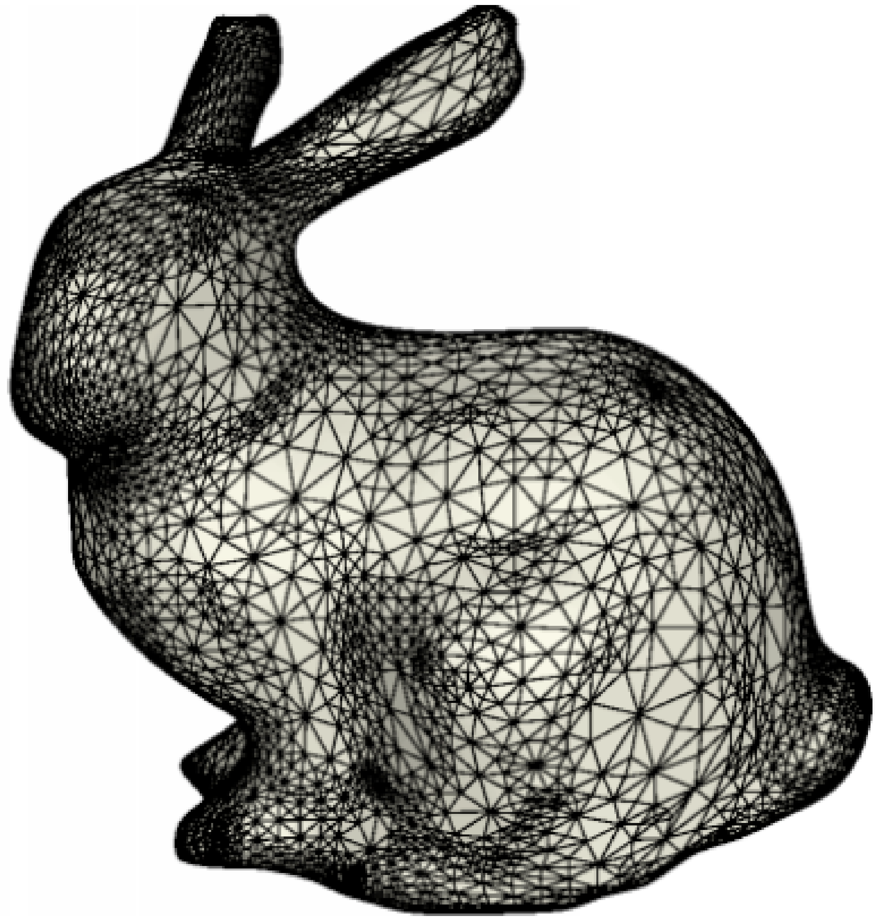


Example 2: Adaptive

- Application-Dependent Criteria



Spatial Selection



<u>Curvature</u>

Conclusions

- Simplicial Multiresolution
  - Powerful Mechanism for Adaptation
- First Part of the Solution for Surface Fitting
  - Simplification
  - Adaptive Refinement
- Second Part (Next)
  - Geodesic Parametrization
  - Bezier Approximation

Adaptive Manifold Fitting (Part II)

Dimas Martínez Morera
UFAL

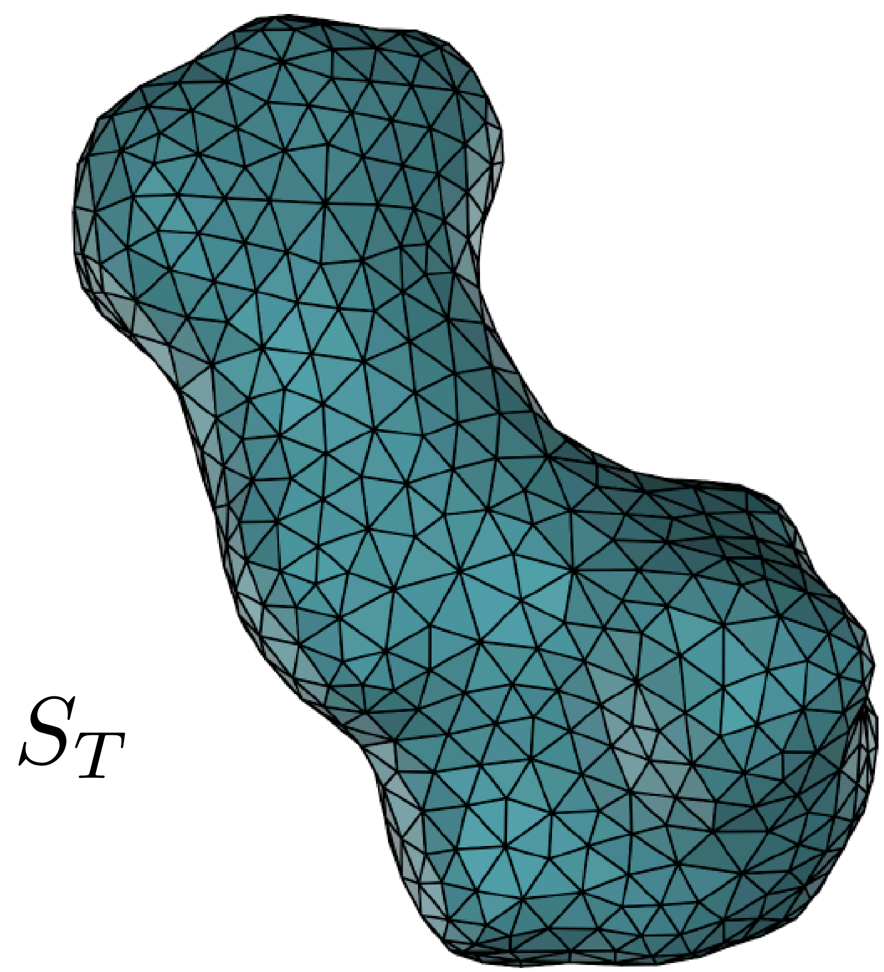
Outline

- The Surface Fitting Problem
- Adaptive Fitting
- Discrete Geodesics
- Conclusions

The Surface Fitting Problem

The Surface Fitting Problem

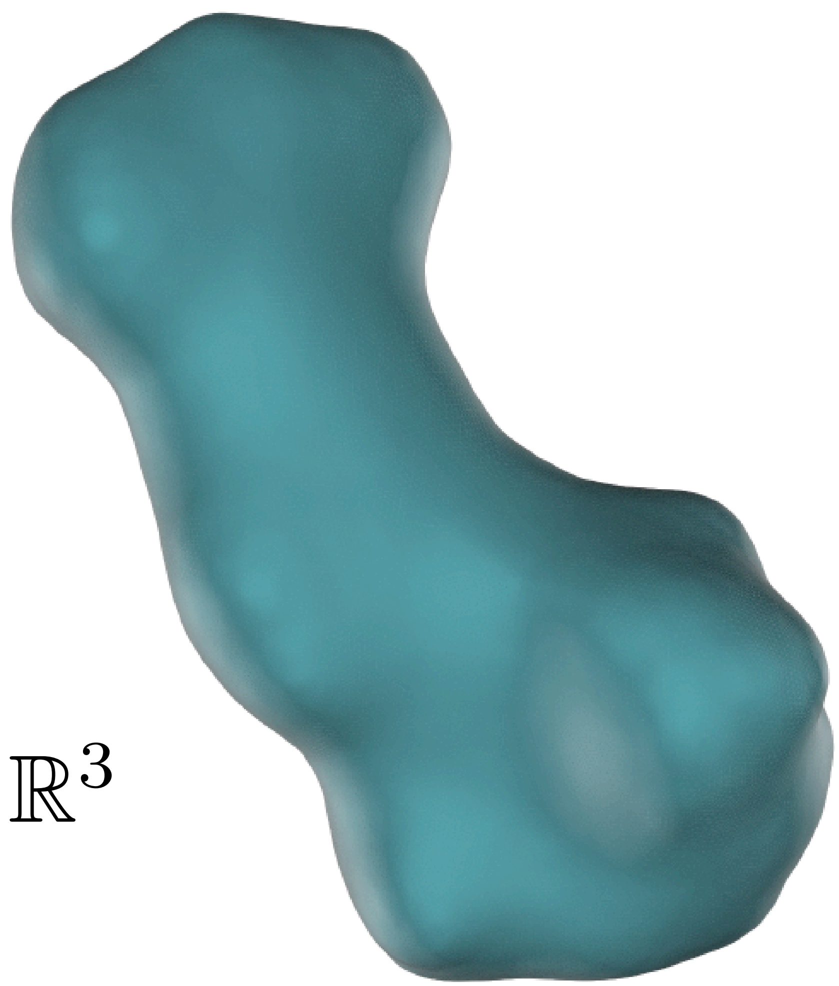
We are a given a piecewise-linear surface, S_T, in $\mathbb{R}^3$, with an empty boundary, a positive integer k, and a positive number ϵ, ...



The Surface Fitting Problem

The Surface Fitting Problem

We want to find a C^k surface $S \subset \mathbb{R}^3$ ...



$S \subset \mathbb{R}^3$

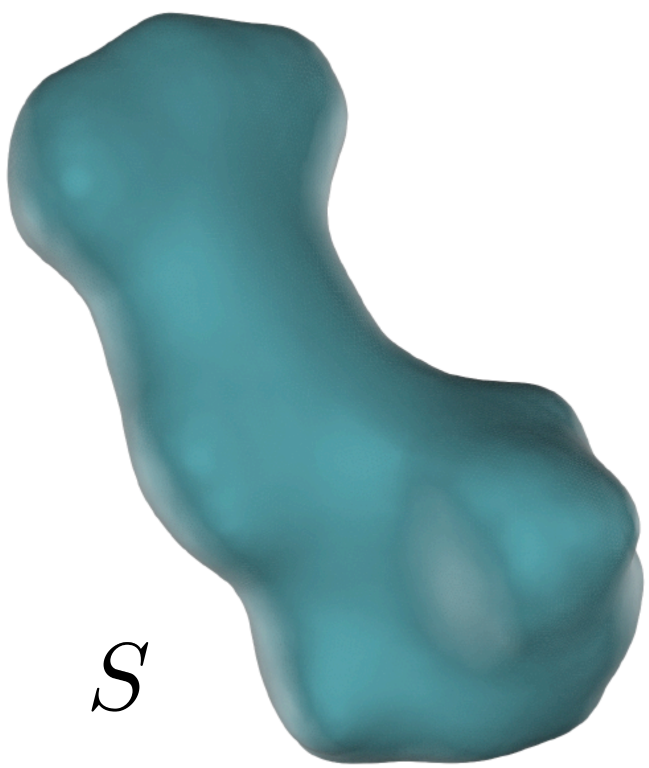
The Surface Fitting Problem

The Surface Fitting Problem

such that there exists a homeomorphism, $h : S \rightarrow |S_T|$, satisfying

$$\|h(v) - v\| \leq \epsilon \,,$$

for every vertex v of S_T.

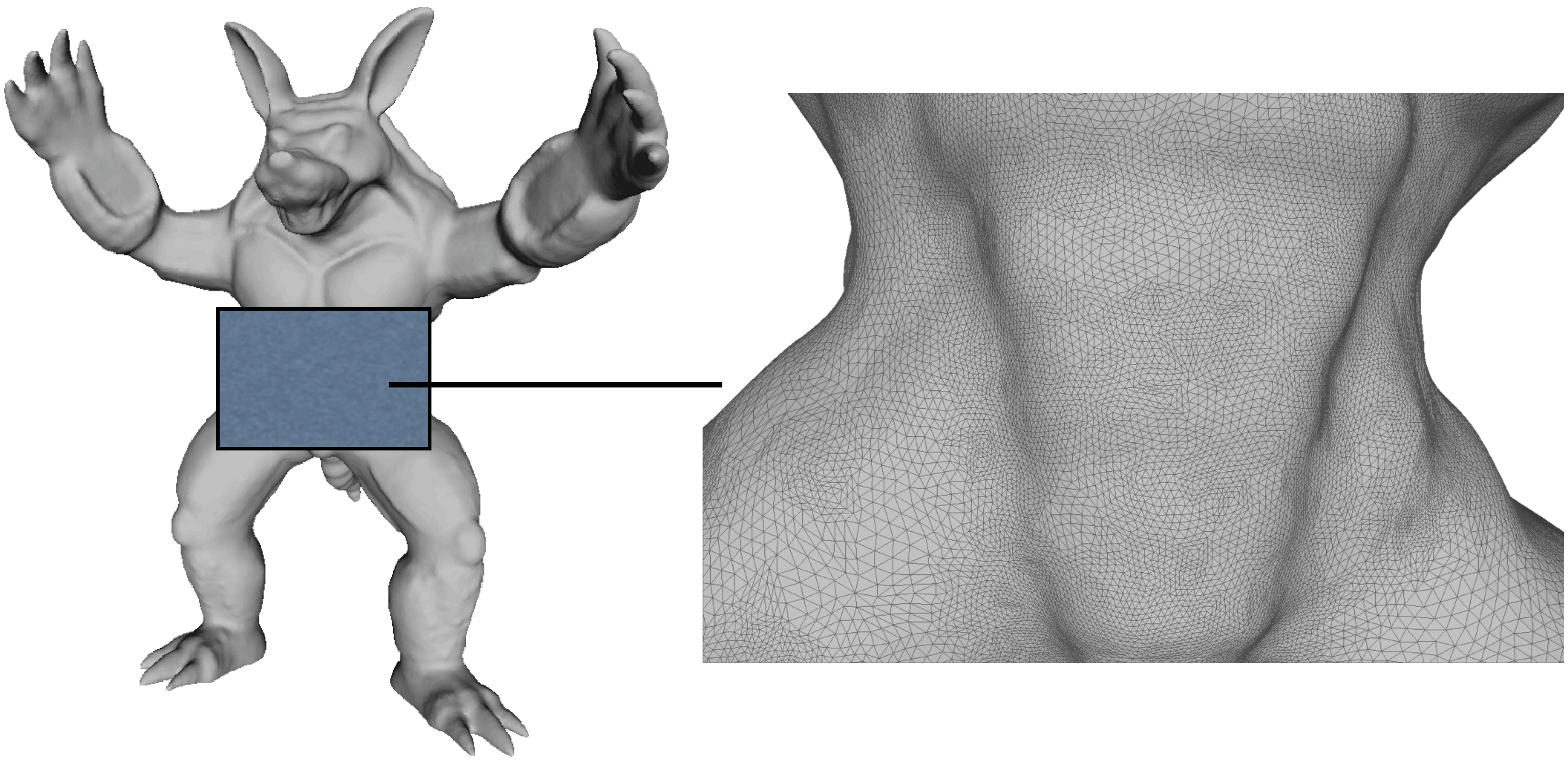


The Surface Fitting Problem

The Surface Fitting Problem

REMARK:

S_T is expected to be "very large" ($\sim 10^6$ vertices).



Adaptive Fitting

Adaptive Fitting

PIPELINE

Adaptive Fitting

S_T

PIPELINE

Adaptive Fitting

$$S_T$$

$$\downarrow$$

$$\boxed{\tilde{S}_T = \text{Simplify } S_T}$$

PIPELINE

Adaptive Fitting

S_T

$\downarrow$

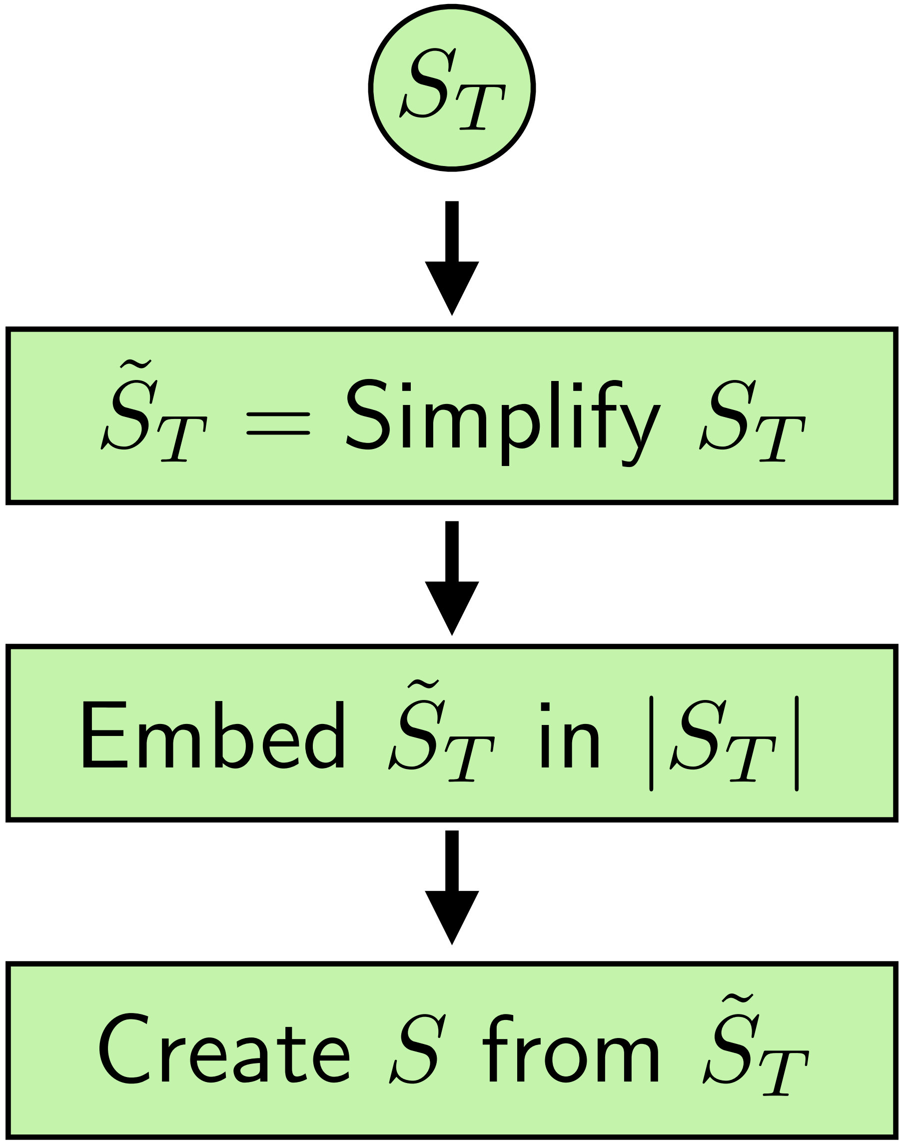
$\tilde{S}_T = \text{Simplify } S_T$

$\downarrow$

Embed $\tilde{S}_T$ in $|S_T|$

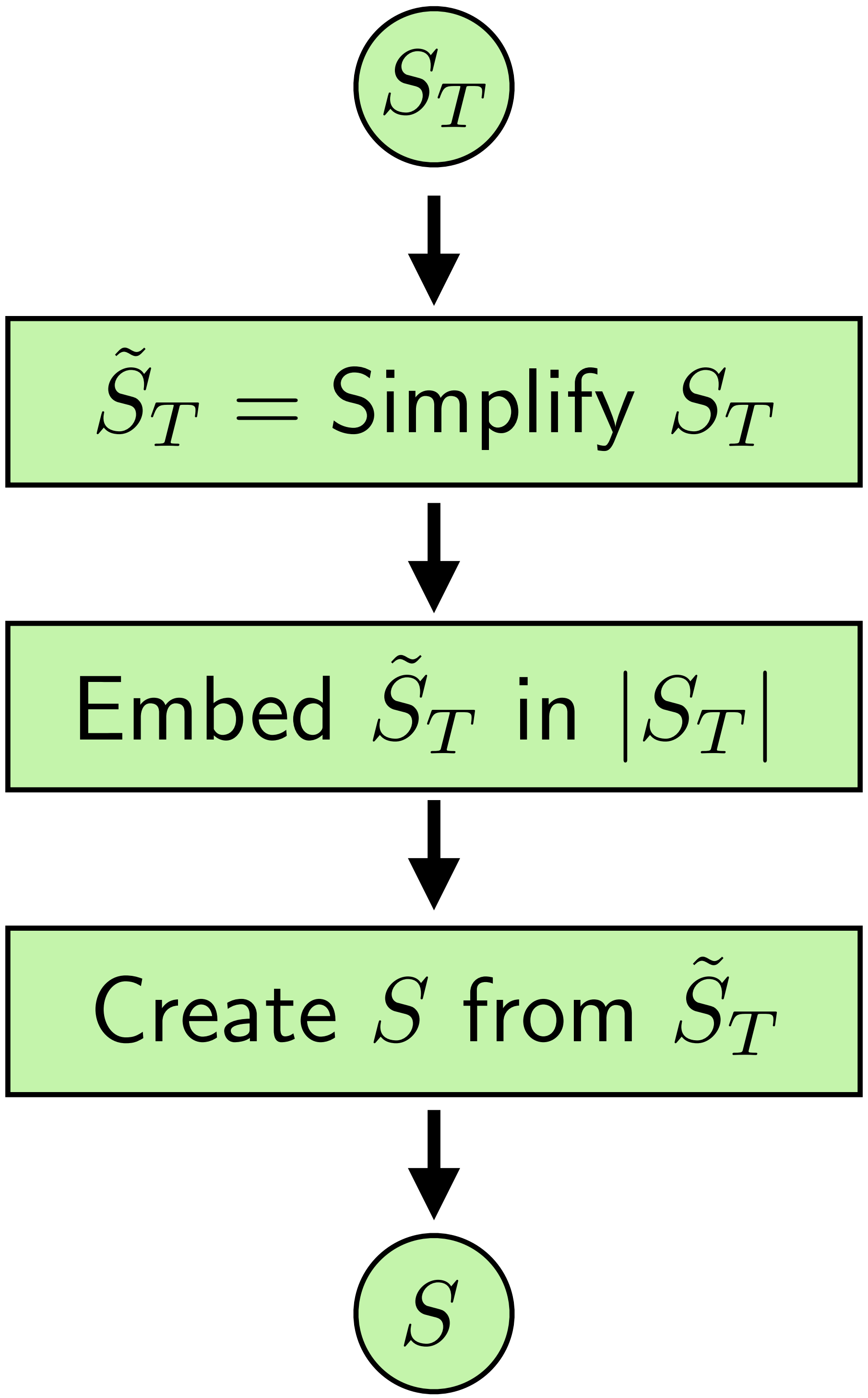
PIPELINE

Adaptive Fitting



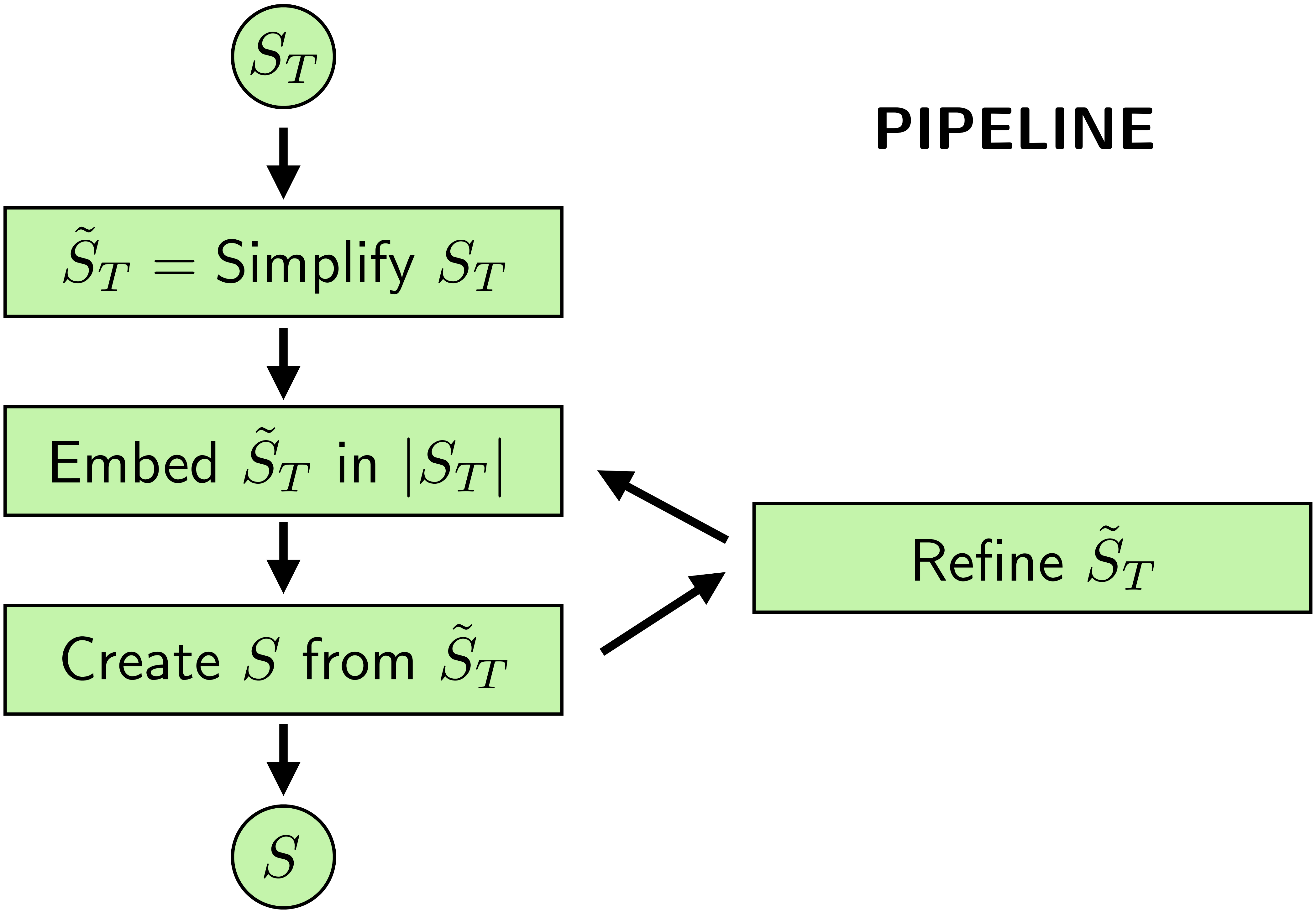
PIPELINE

Adaptive Fitting



PIPELINE

Adaptive Fitting



PIPELINE

Adaptive Fitting

$S_T \longrightarrow \boxed{\tilde{S}_T = \text{Simplify } S_T}$

Adaptive Fitting

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- Four-Face Clusters Algorithm

Adaptive Fitting

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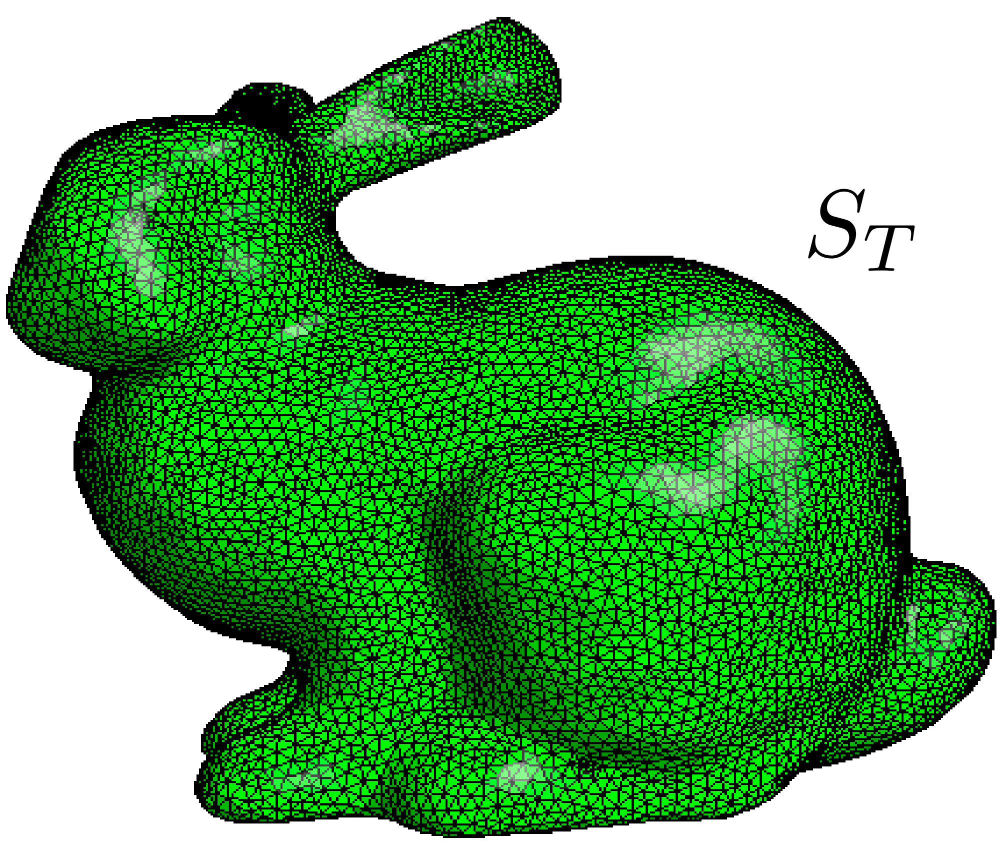
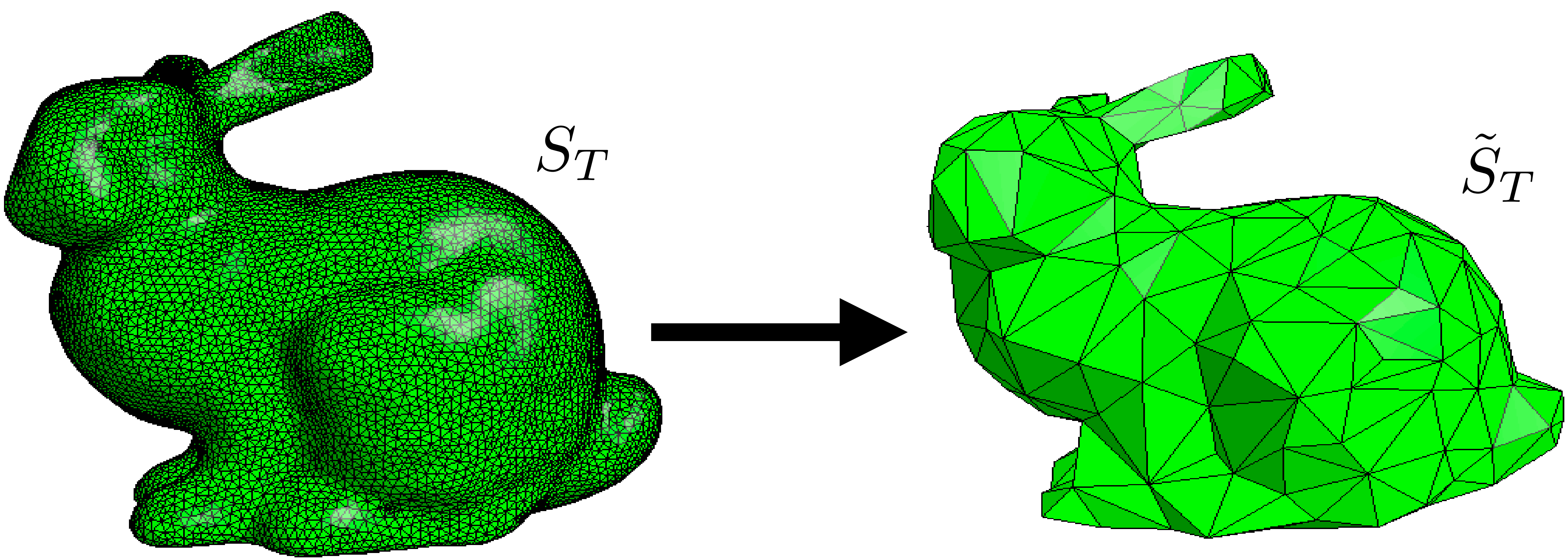


Adaptive Fitting

$$S_T \longrightarrow \boxed{\tilde{S}_T = \text{Simplify } S_T}$$

- Four-Face Clusters Algorithm



Adaptive Fitting

Adaptive Fitting

Embed $\tilde{S}_T$ in $|S_T|$

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- Each edge of $\tilde{S}_T$ is embedded in $|S_T|$ as a “geodesic”.

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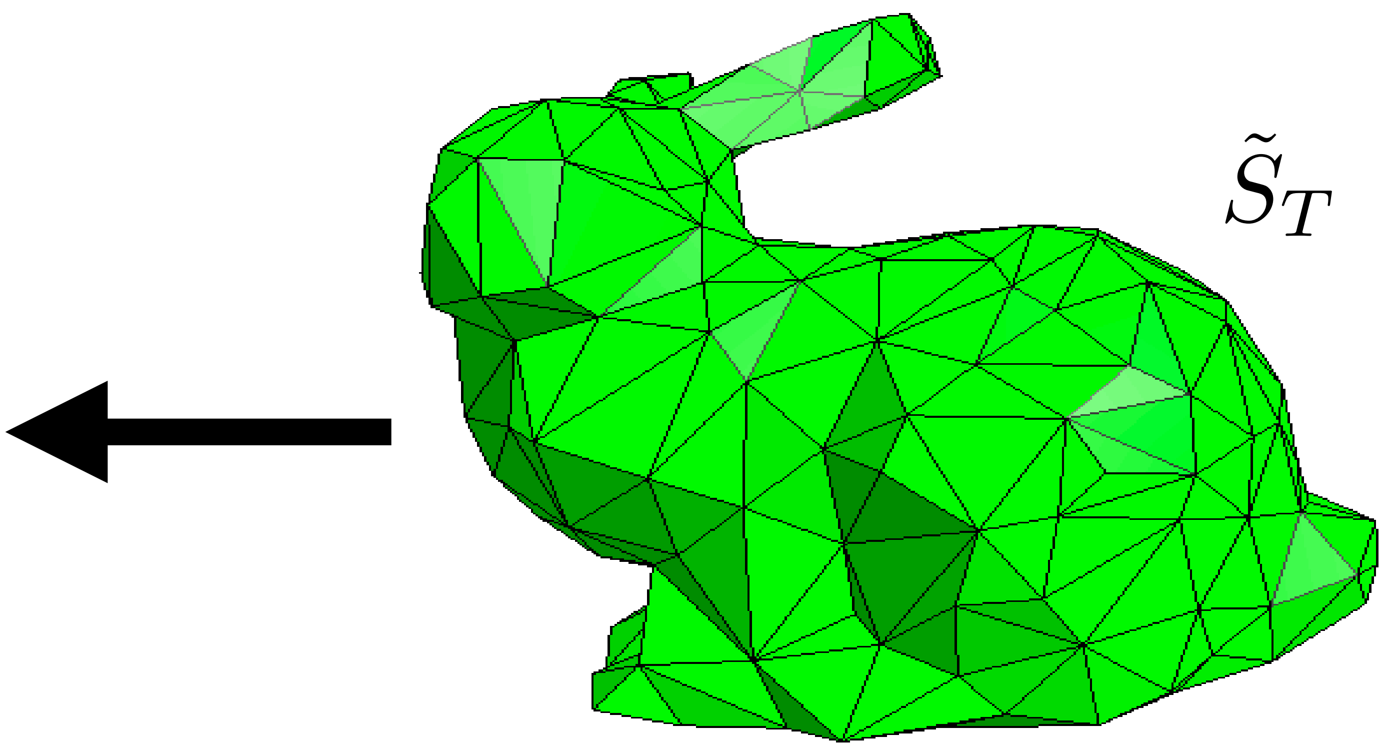

Adaptive Fitting

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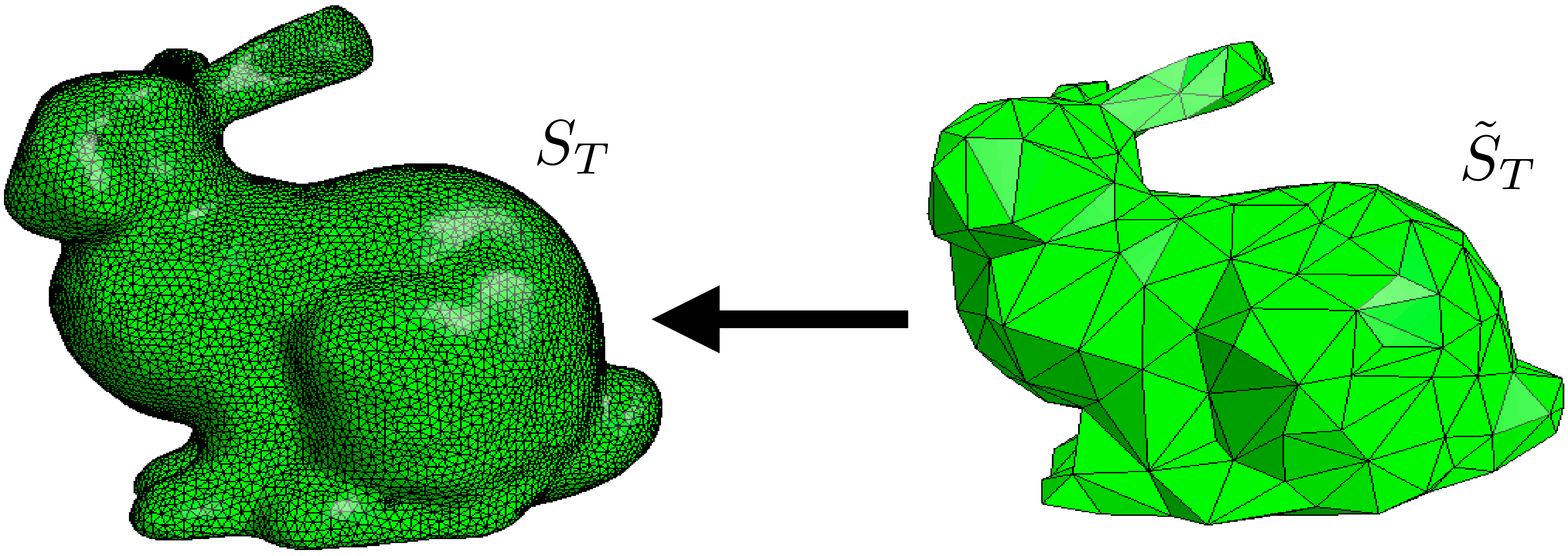

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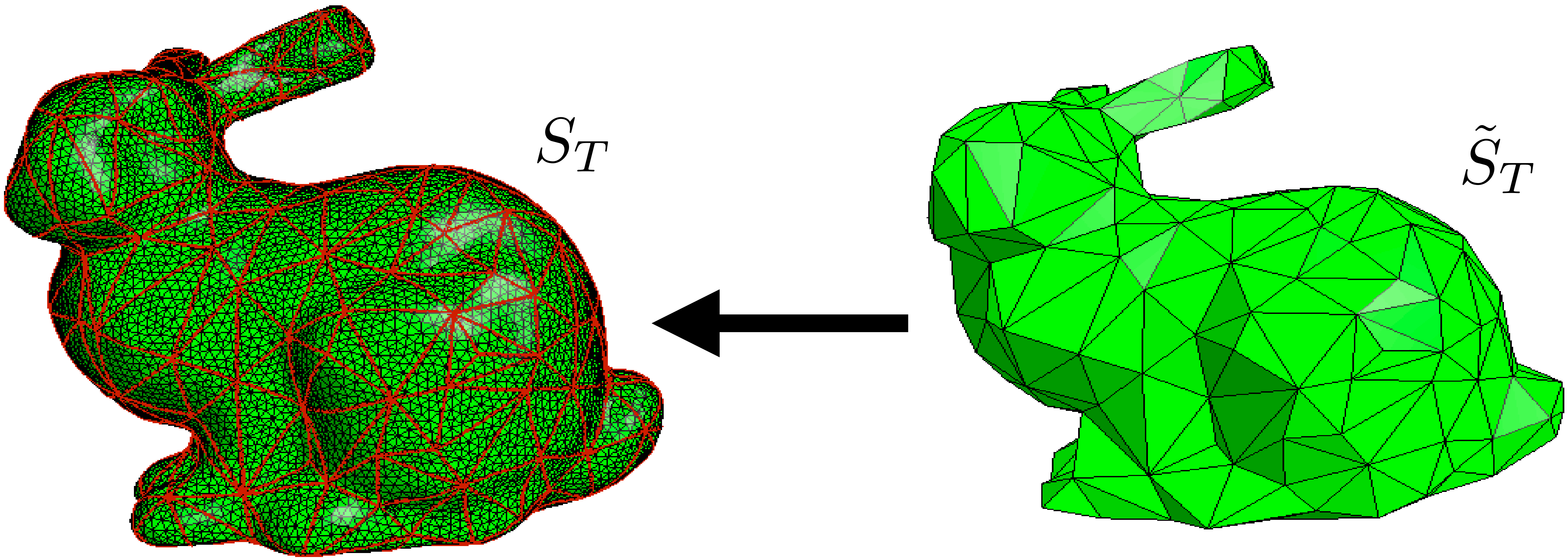
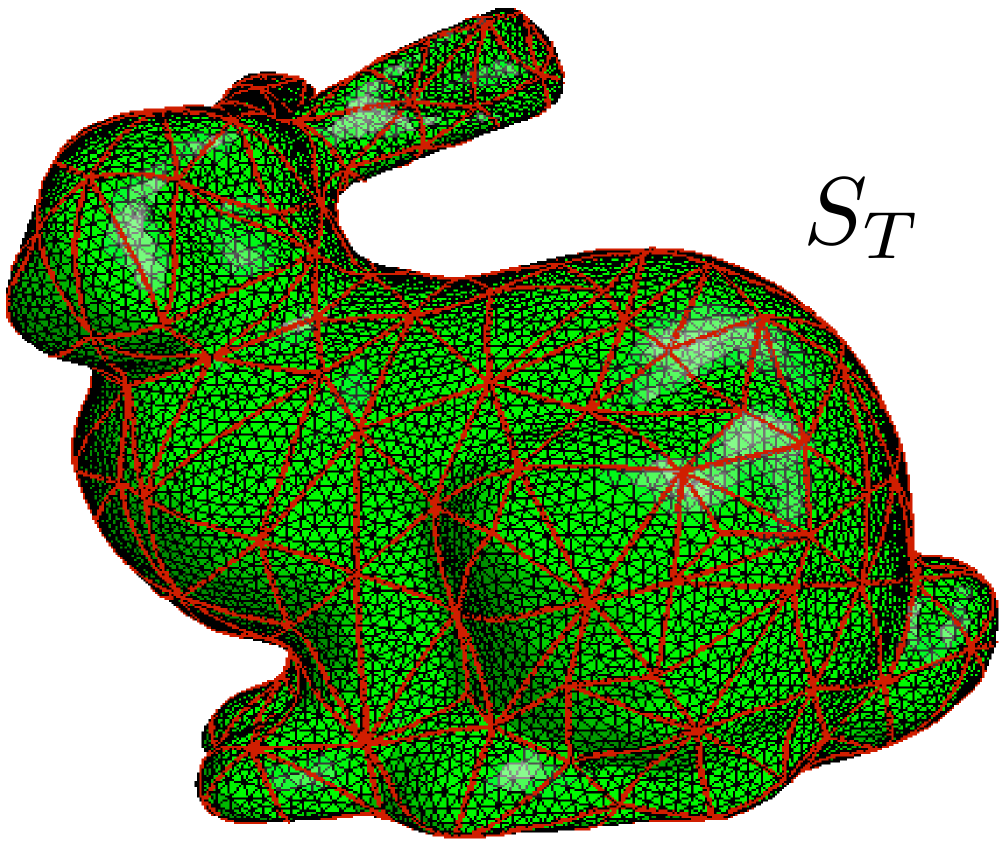


Adaptive Fitting

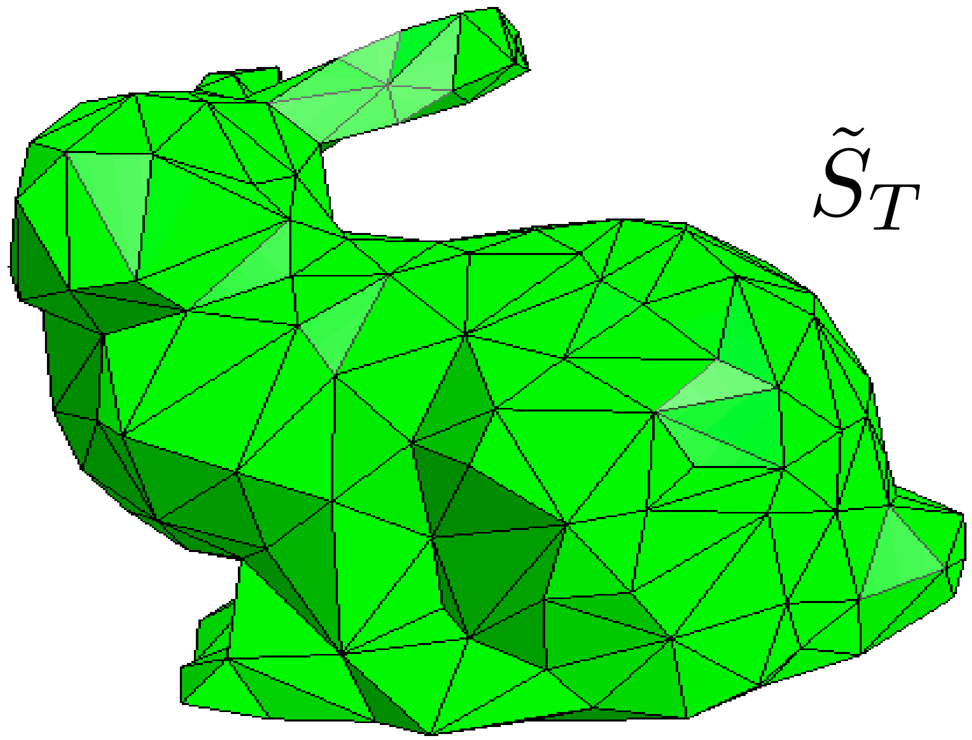
Adaptive Fitting

REMARK:

The vertices of $\tilde{S}_T$ ARE vertices of S_T.



S_T



$\tilde{S}_T$

Adaptive Fitting

REMARK:

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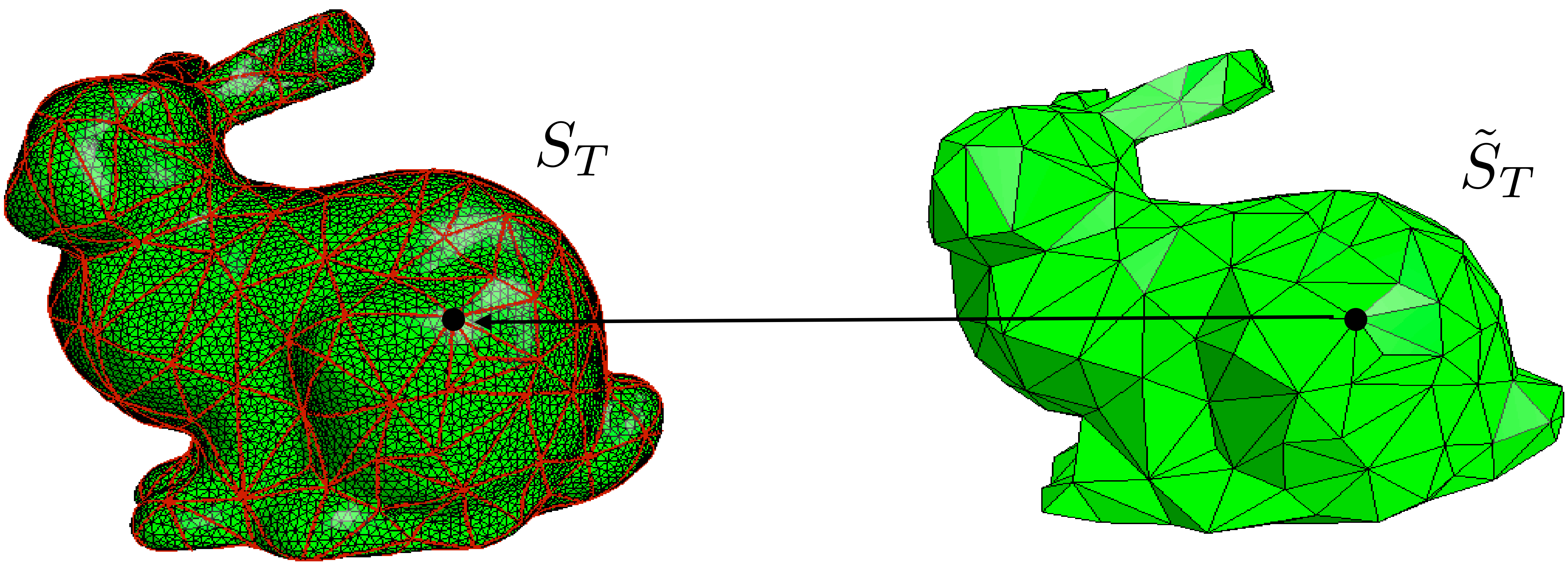

Adaptive Fitting

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PROBLEM:

When defining geodesic triangles, we can violate the manifold property of the geodesic mesh, as illustrated by the figure below:

Adaptive Fitting

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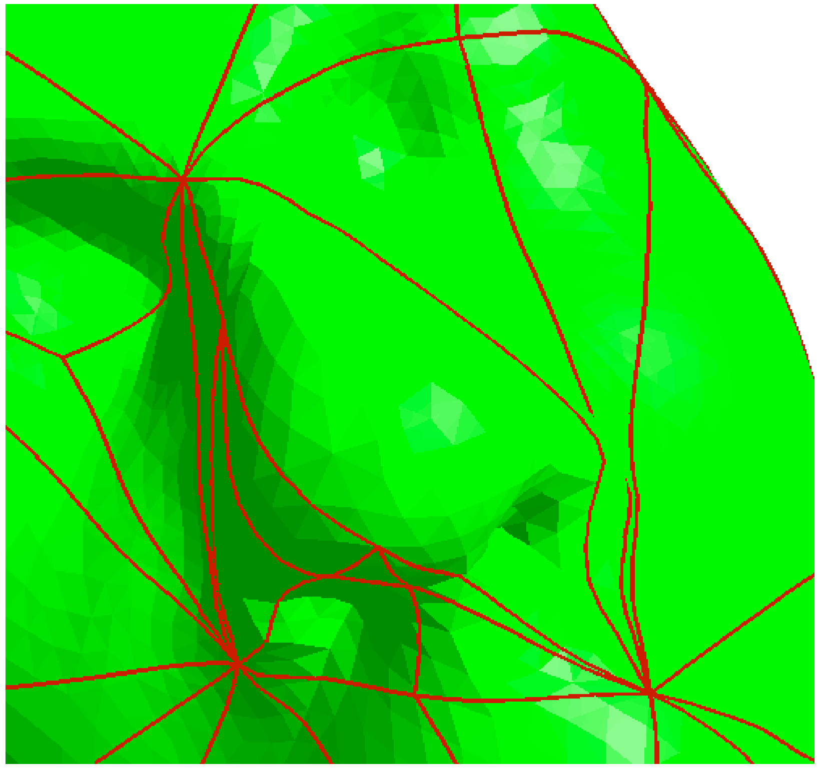
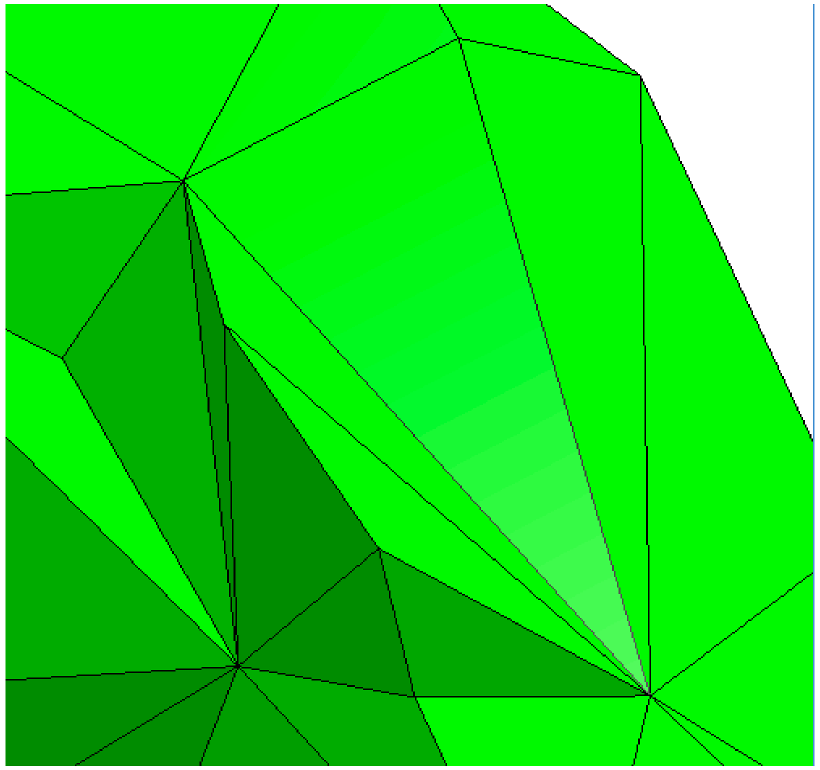




Adaptive Fitting

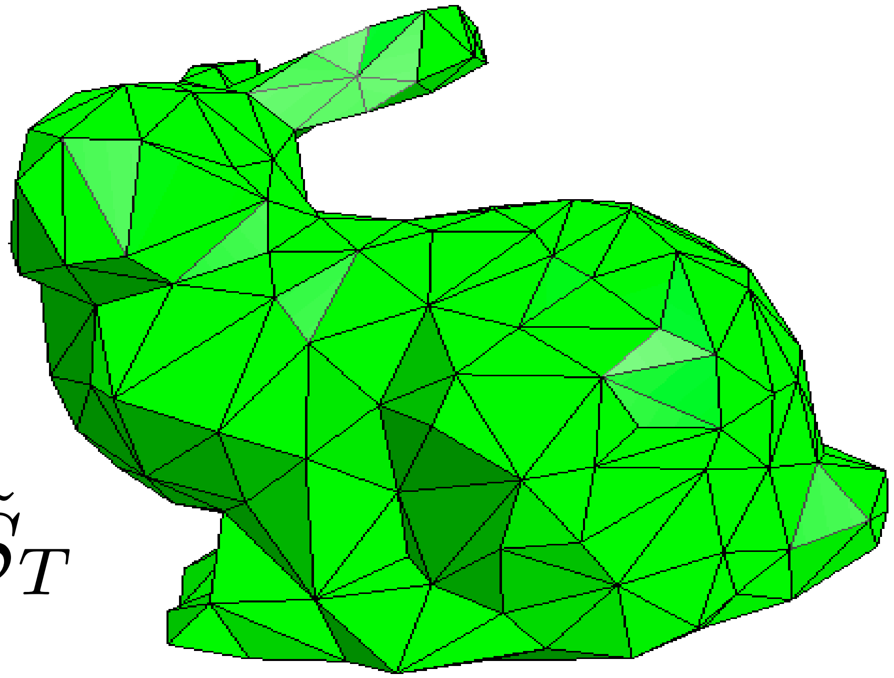
Adaptive Fitting

Create S from $\tilde{S}_T$

Adaptive Fitting

Create S from $\tilde{S}_T$

- For each vertex v of $\tilde{S}_T$, we consider the P-polygon, P_v, of v in $\mathbb{R}^2$, and the standard triangulation, T_v, of the P-polygon P_v.



$\tilde{S}_T$

Adaptive Fitting

Create S from $\tilde{S}_T$

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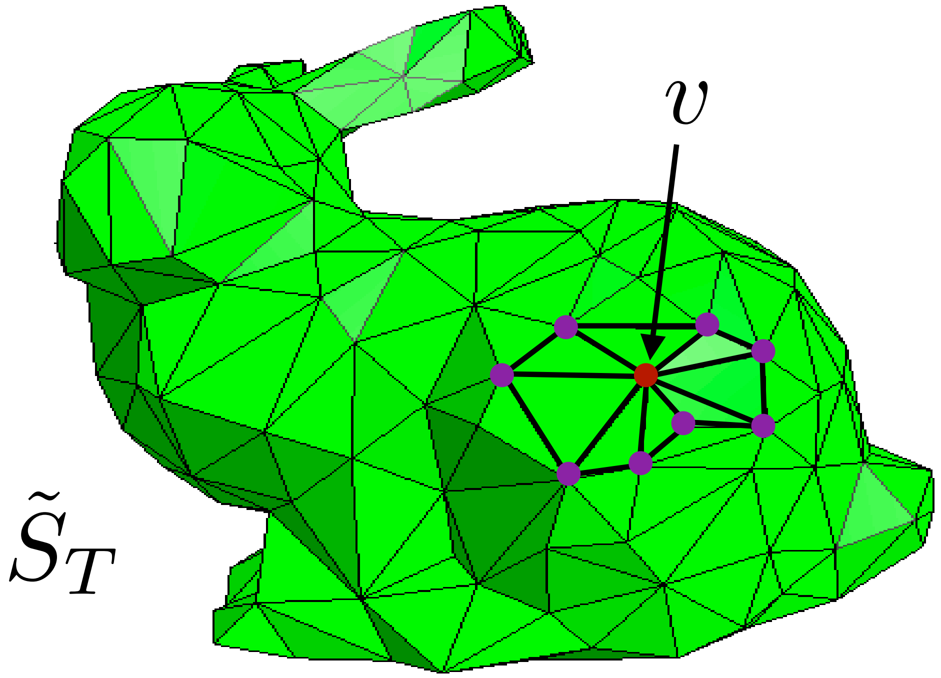

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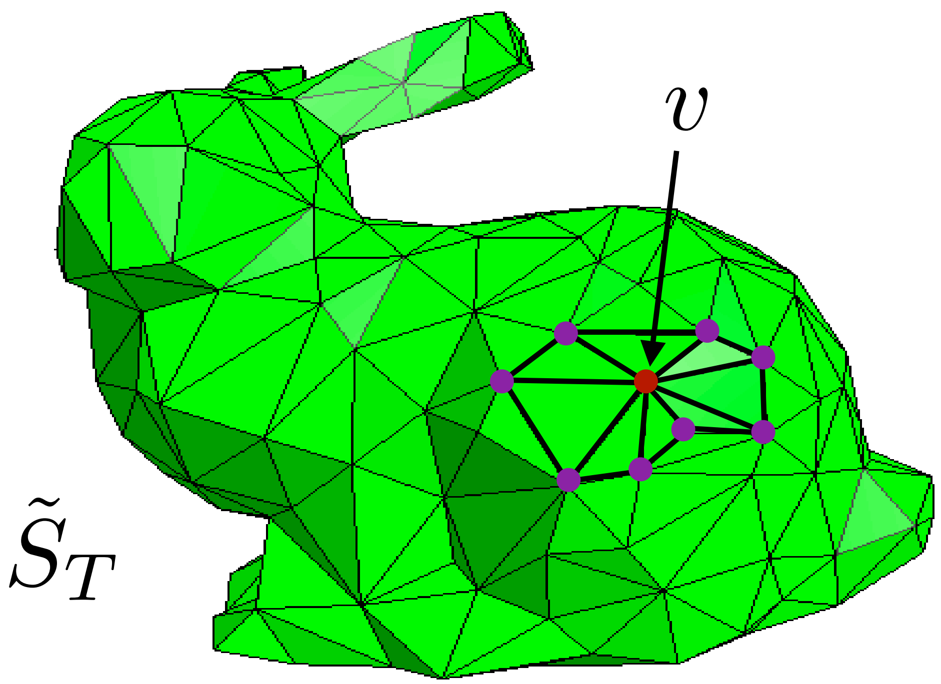
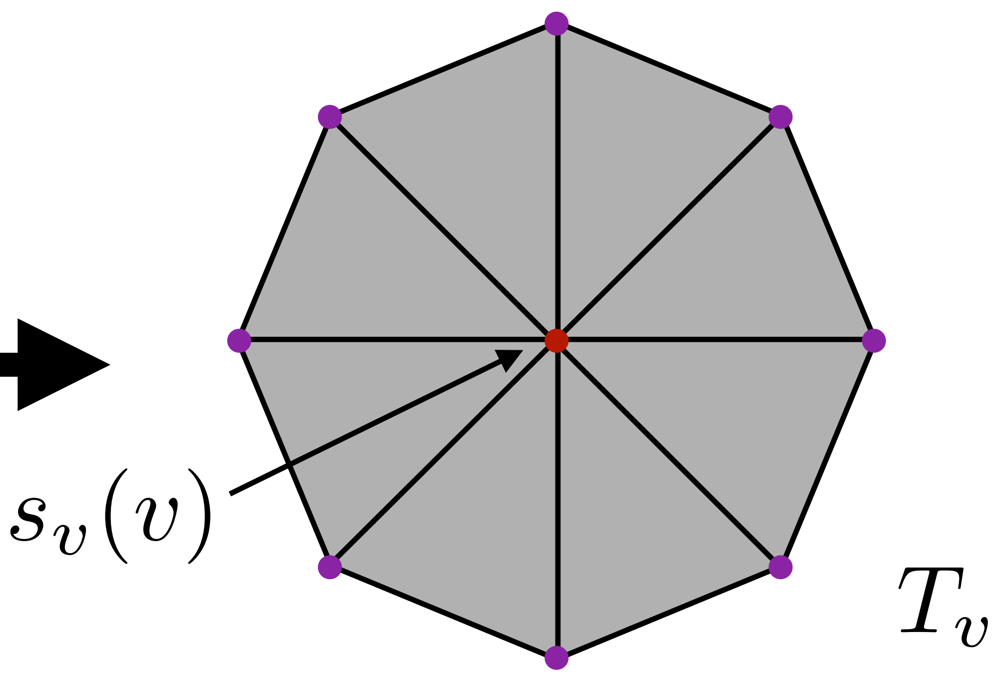




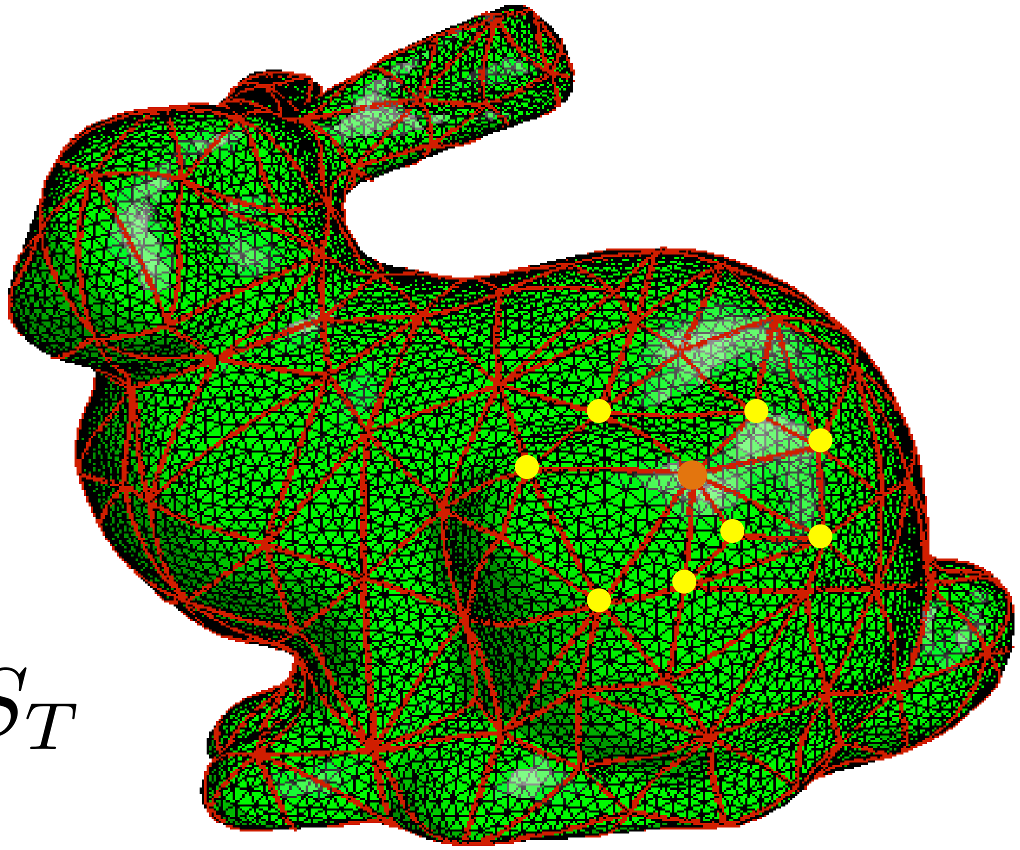
Adaptive Fitting

Create S from $\tilde{S}_T$

Adaptive Fitting

Create S from $\tilde{S}_T$

- Consider the embedding of the star, $st(v, \tilde{S}_T)$, of v in S_T.



S_T

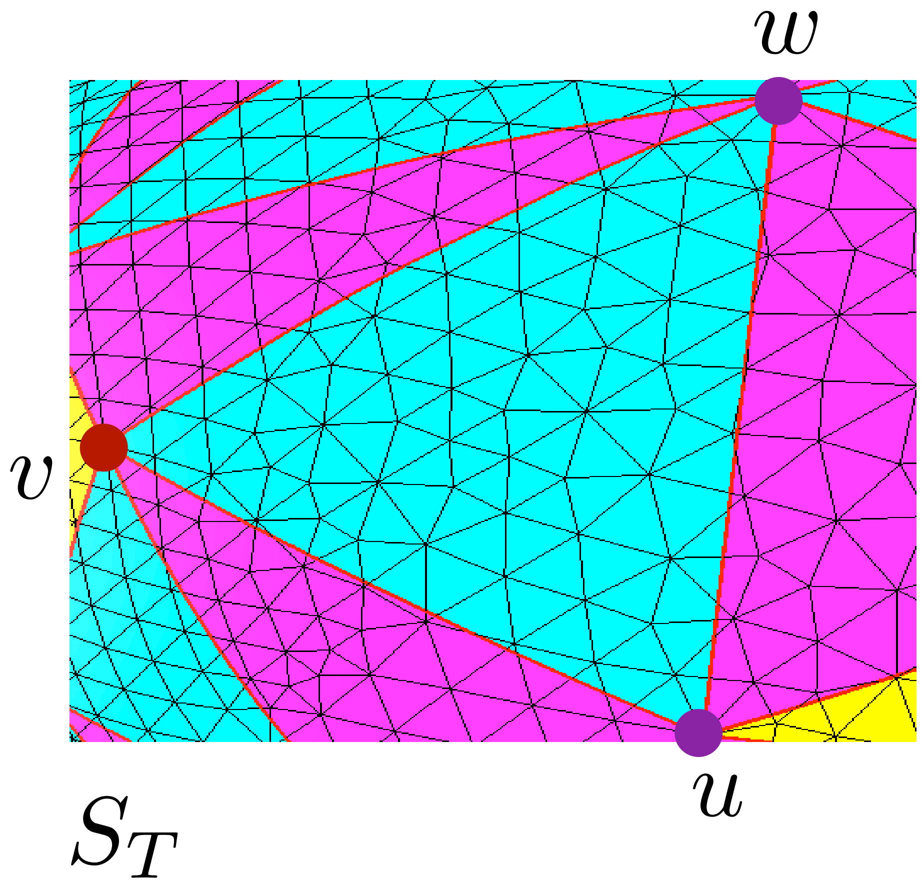
Adaptive Fitting

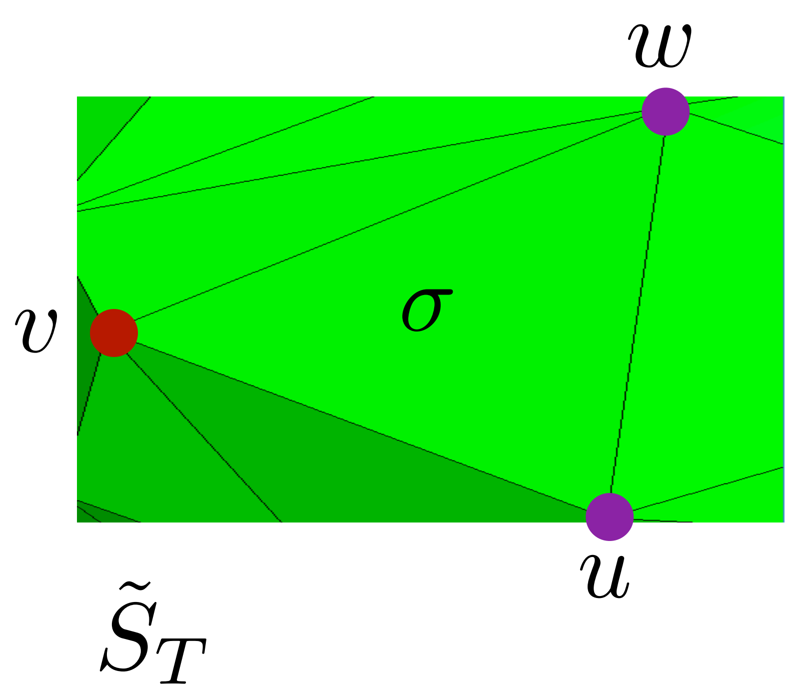
Create S from $\tilde{S}_T$

Adaptive Fitting

Create S from $\tilde{S}_T$

- Map the vertices of S_T bounded by the embedding of $st(v, \tilde{S}_T)$ to T_v.





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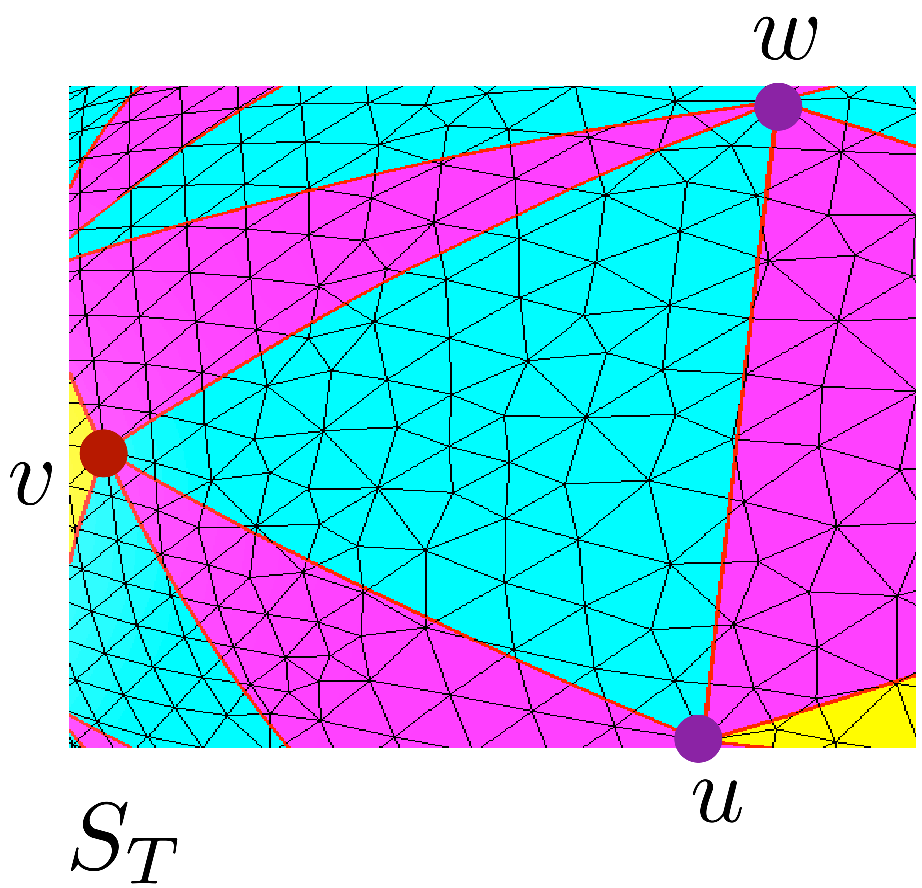

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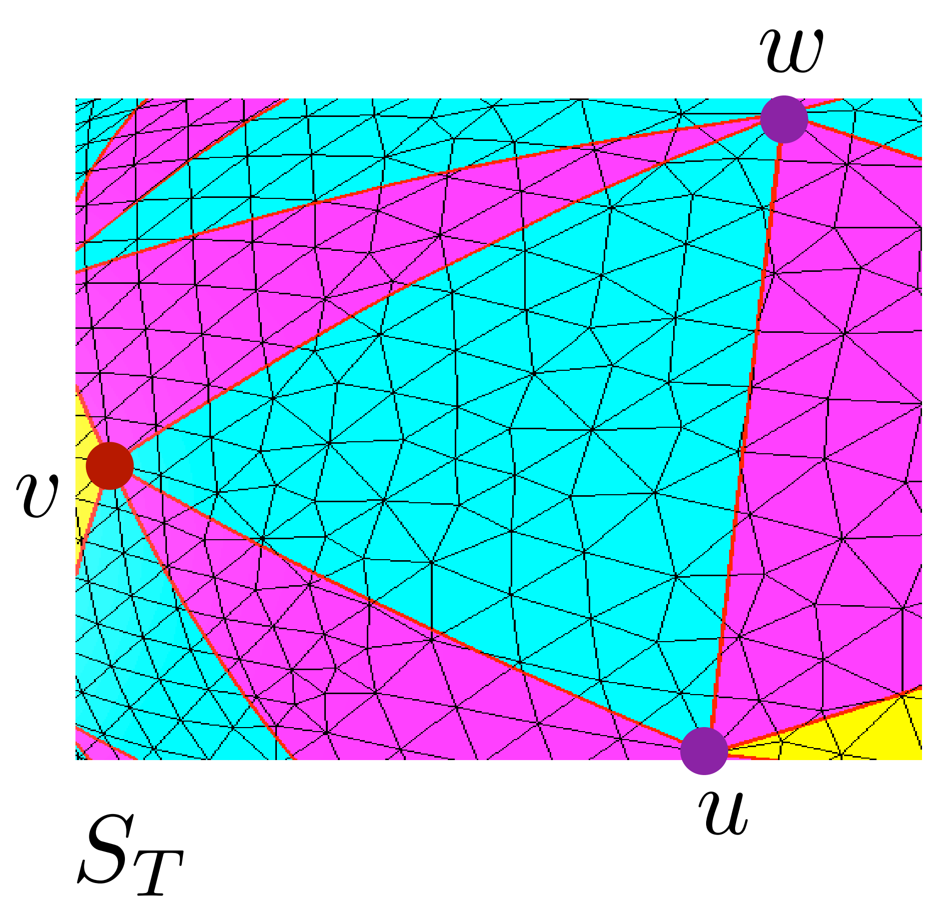

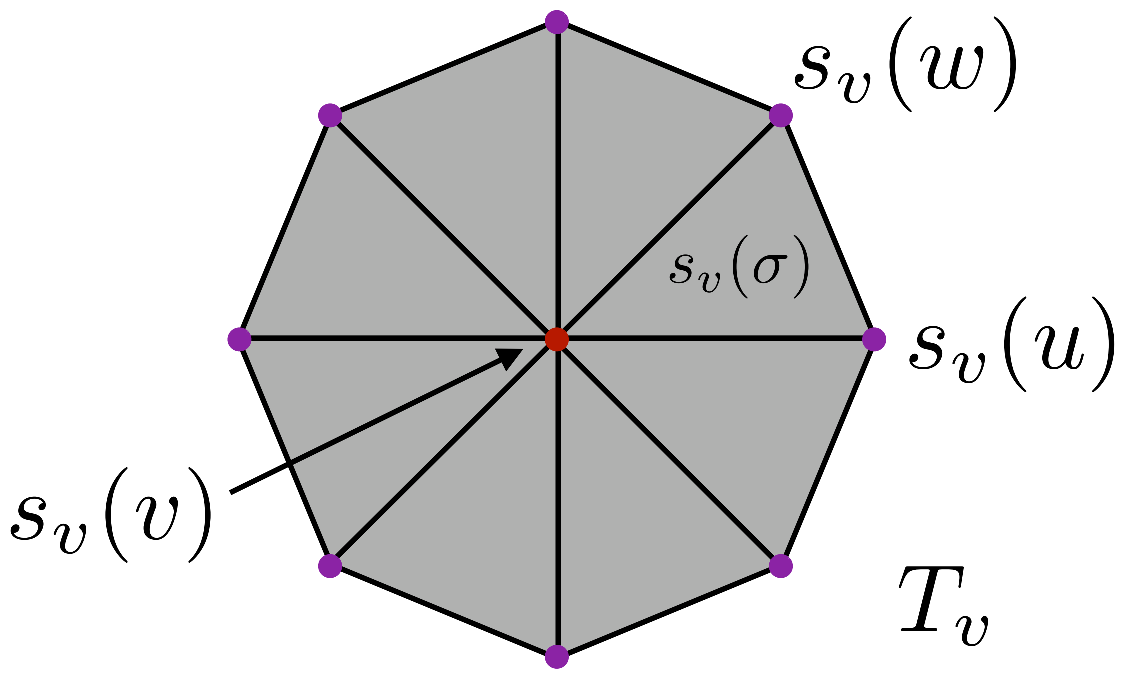

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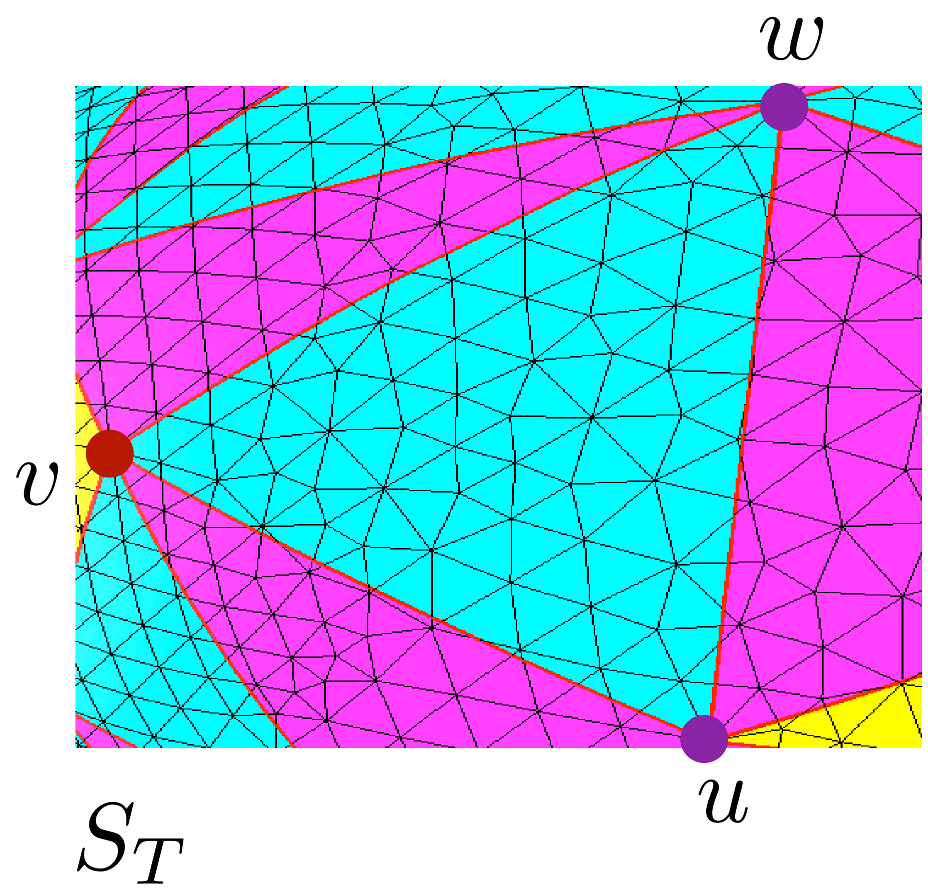

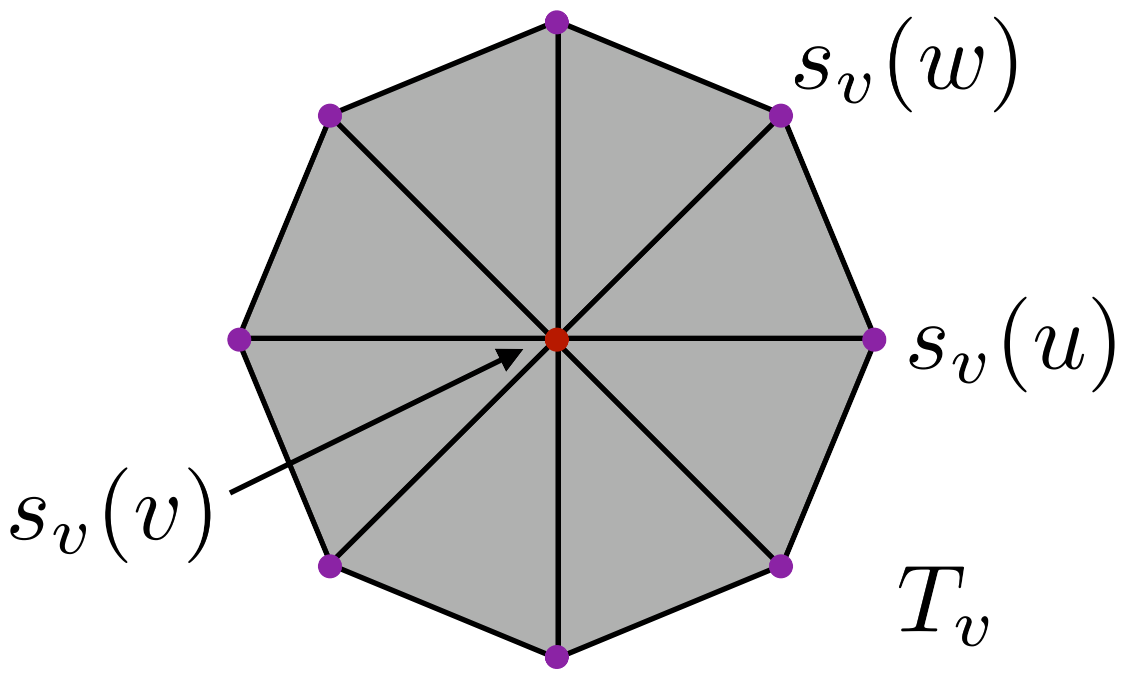

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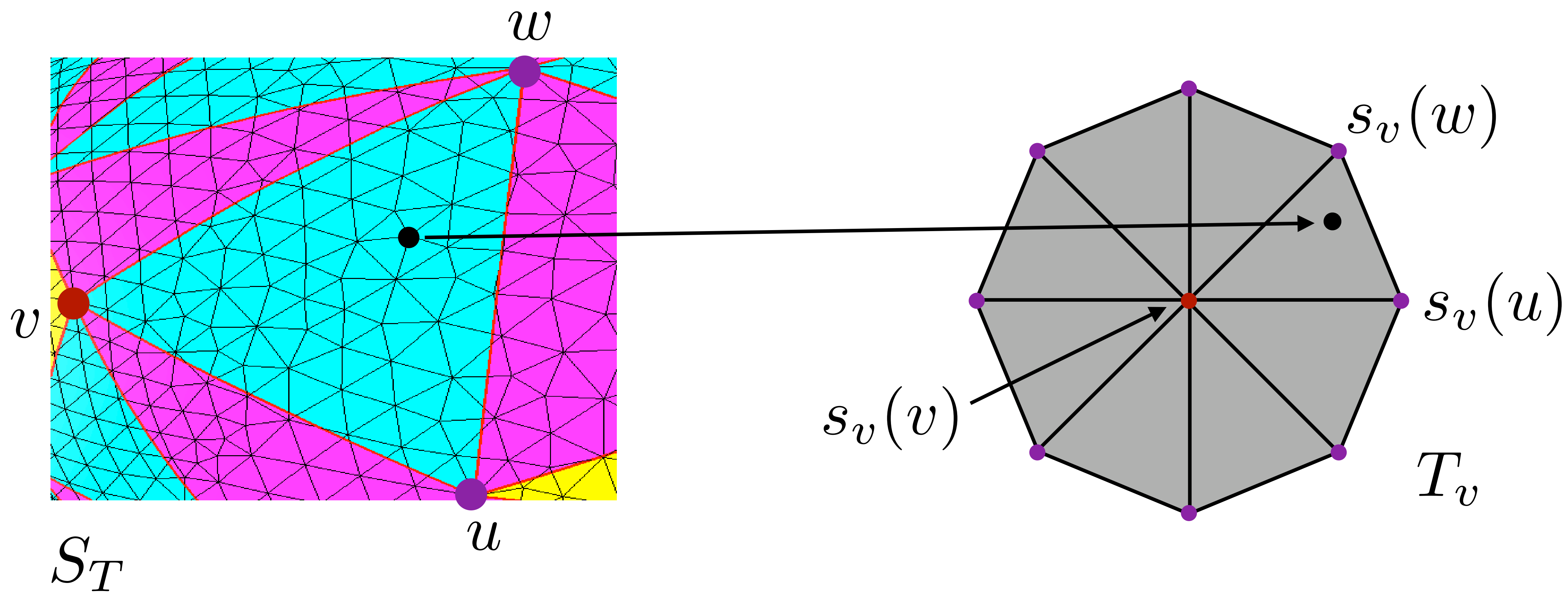
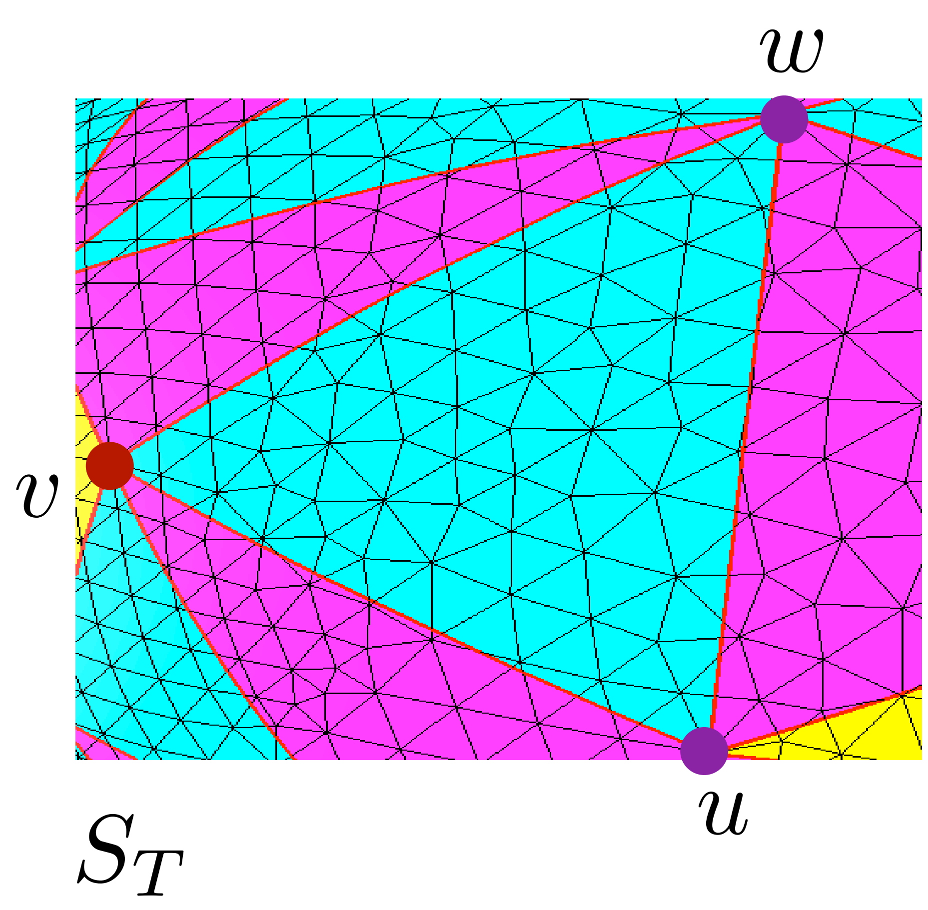
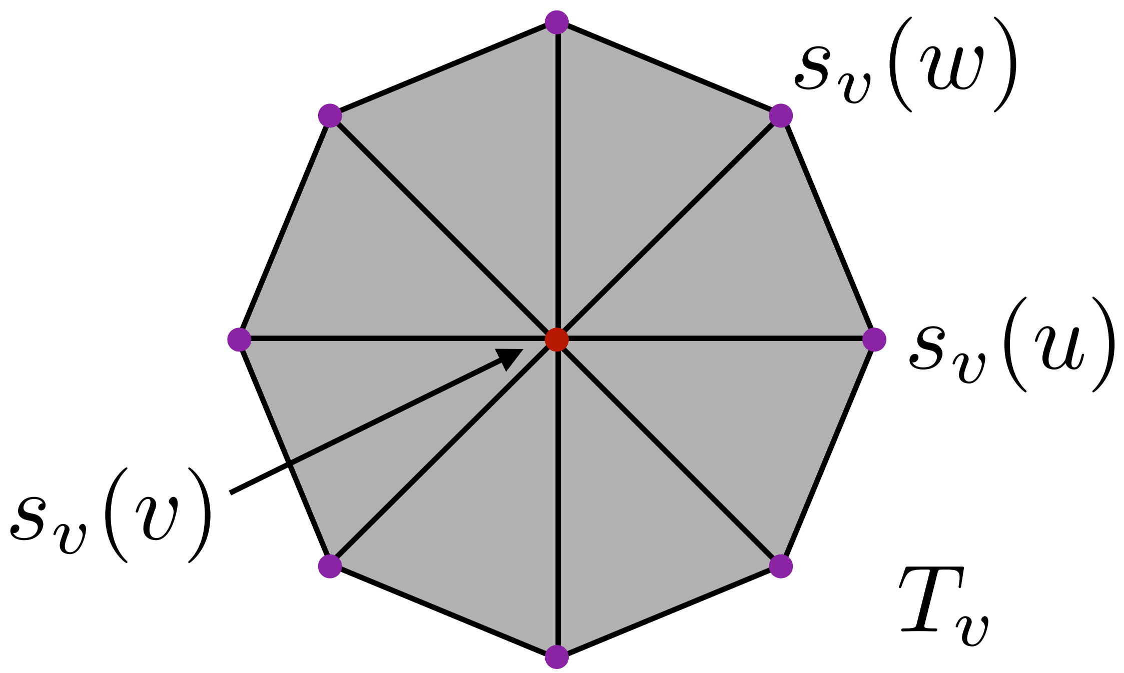


Adaptive Fitting

Create S from $\tilde{S}_T$

- Points where geodesics intersect edges of S_T are also mapped to T_v.





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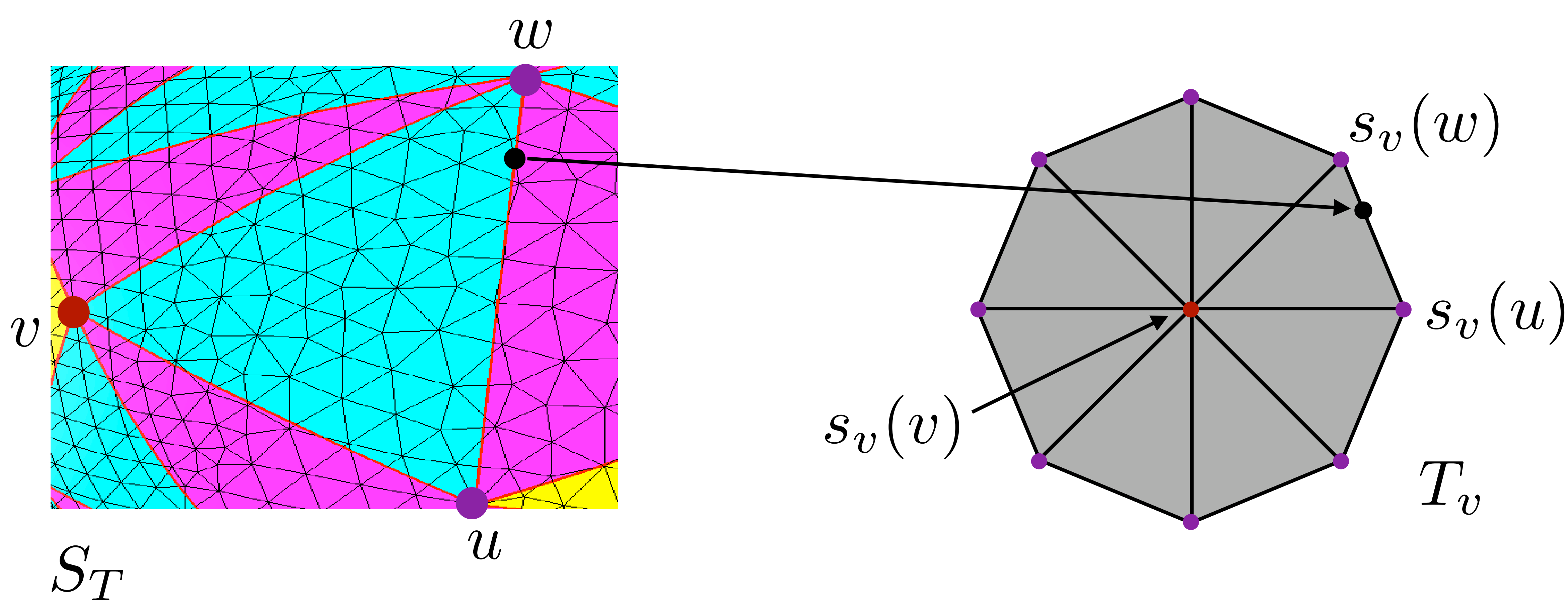


Adaptive Fitting

Create S from $\tilde{S}_T$

Adaptive Fitting

Create S from $\tilde{S}_T$

- How is this mapping done?

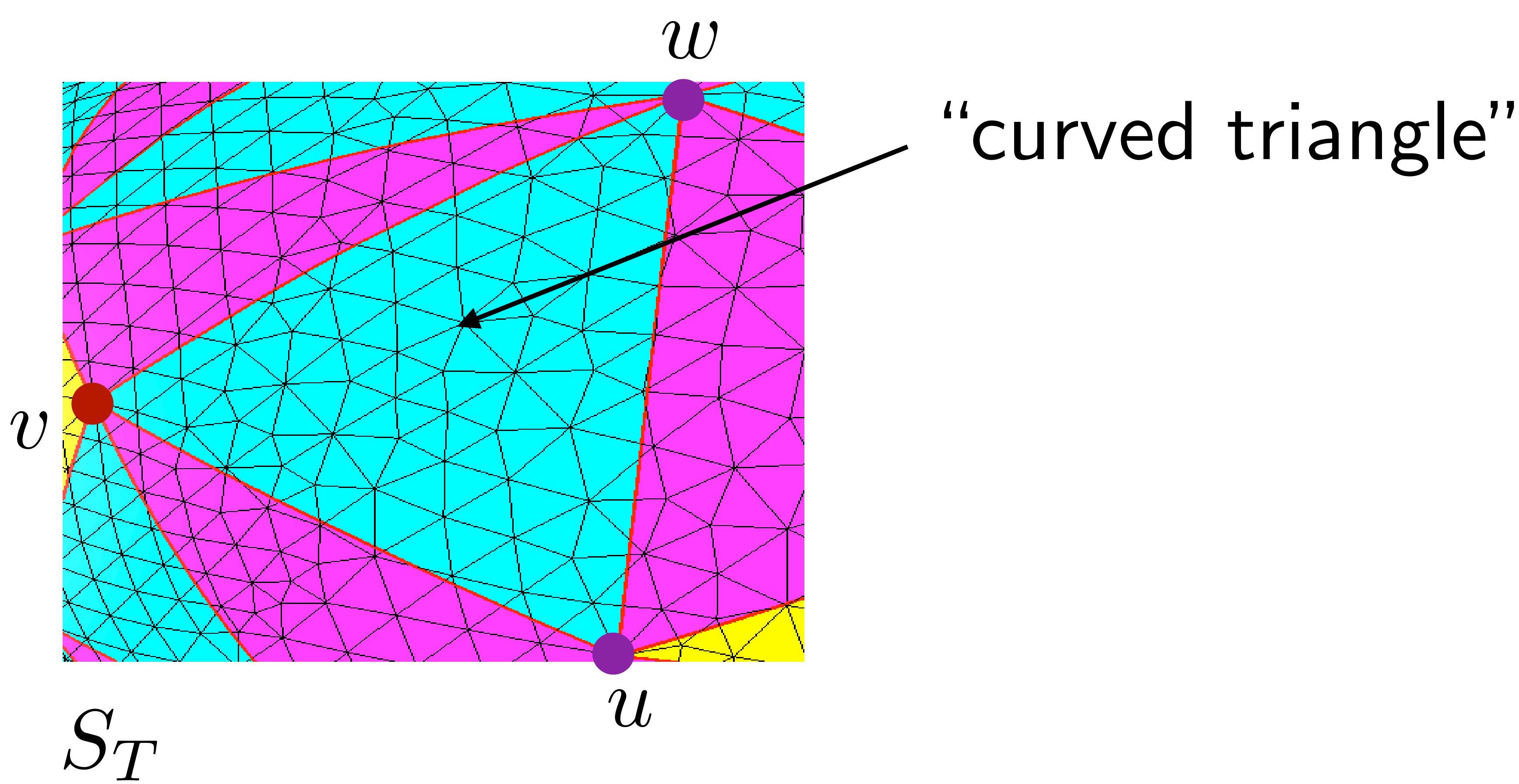
Adaptive Fitting

Create S from $\tilde{S}_T$

Adaptive Fitting

Create S from $\tilde{S}_T$

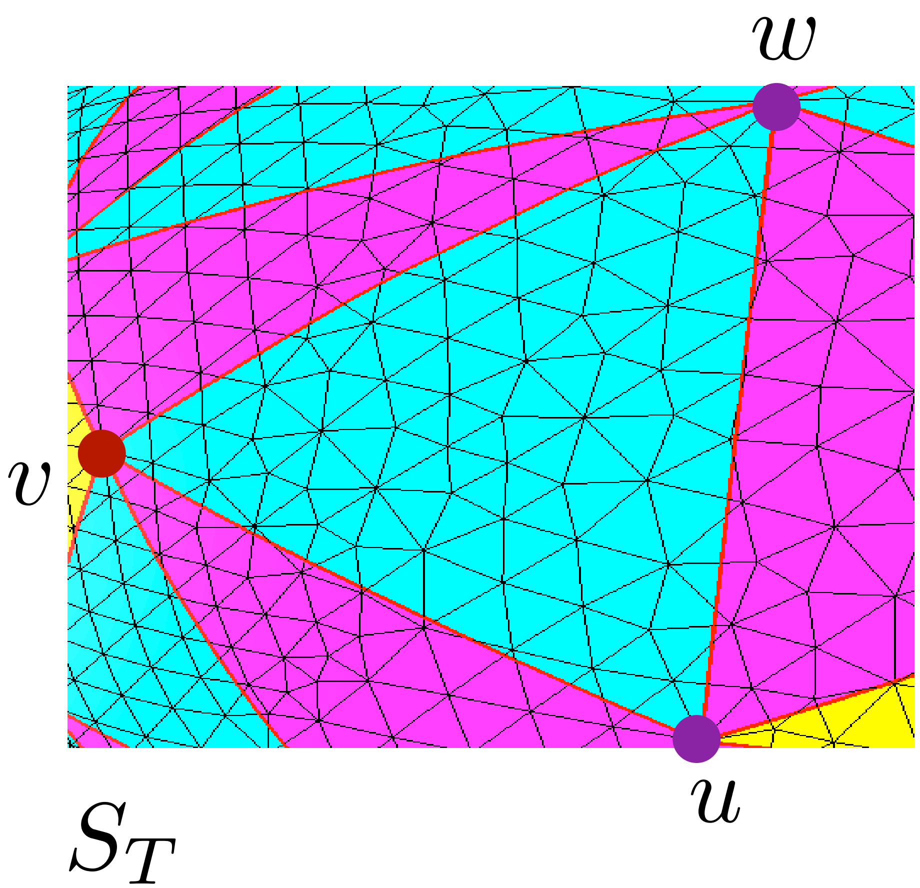
- We map the vertices in each "curved" triangle separately.



Adaptive Fitting

Create S from $\tilde{S}_T$

- We use Floater's parametrization to build the map for each "curved" triangle.



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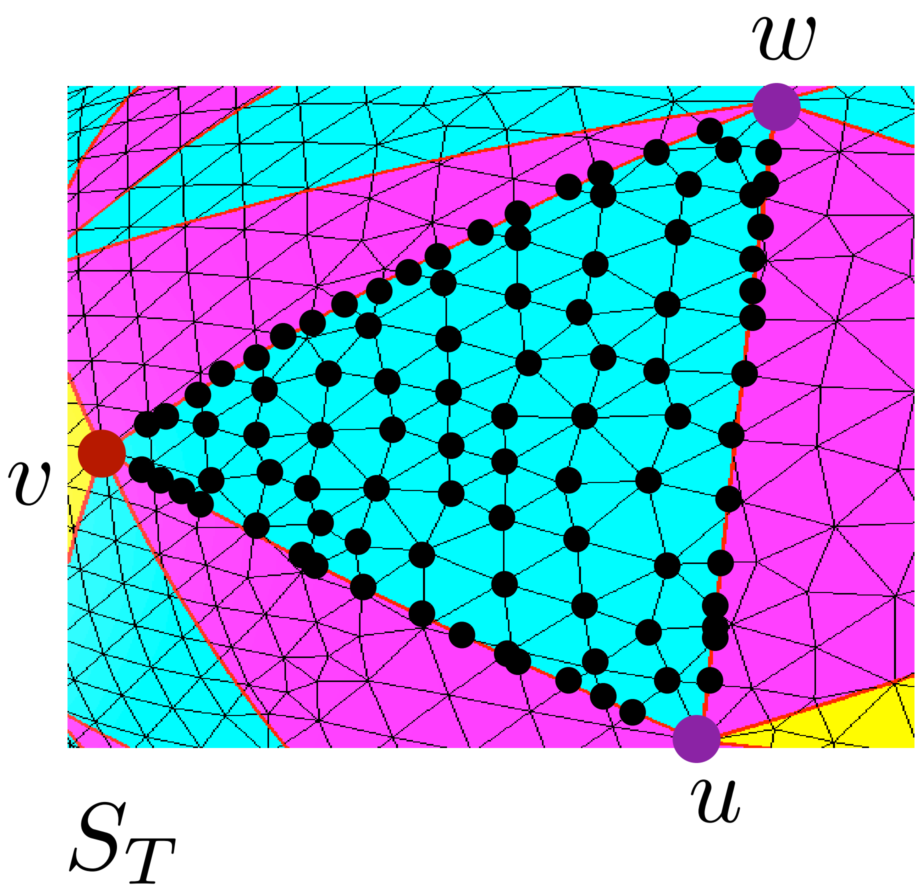

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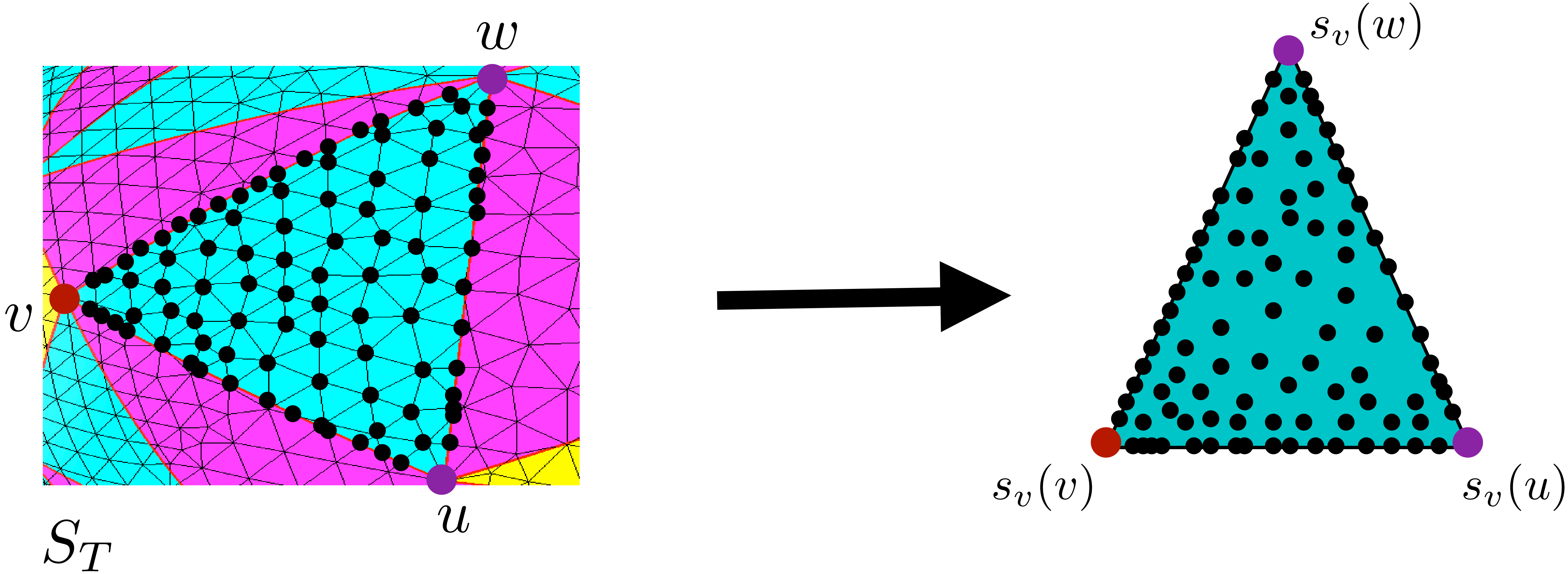
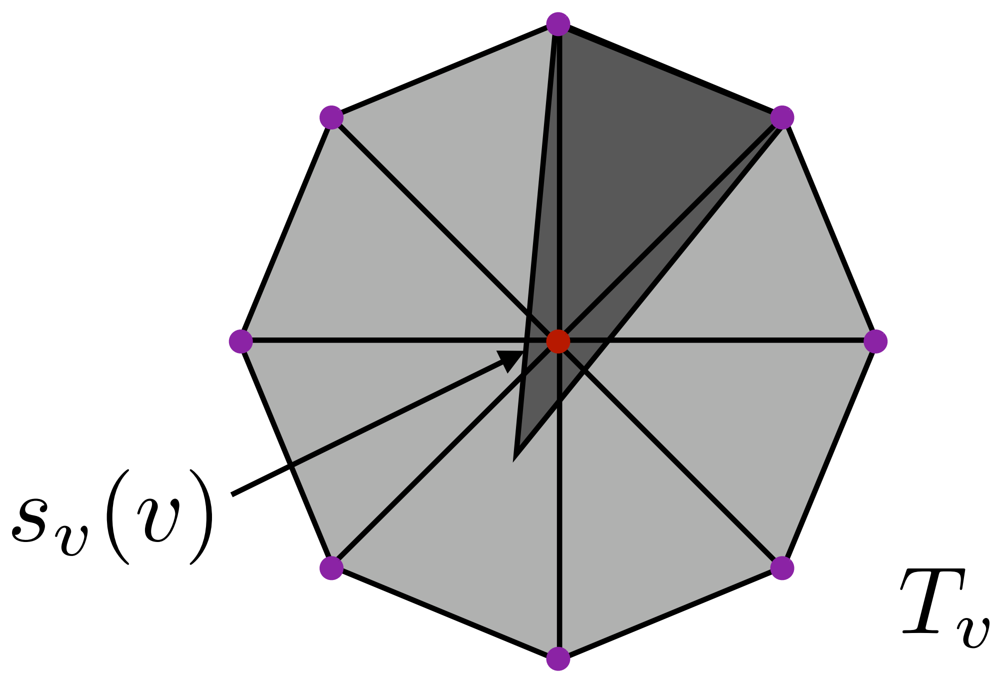


Adaptive Fitting

Create S from $\tilde{S}_T$

- For each triangle in $st(v, \tilde{S}_T)$, compute the shape function $\psi_{(\sigma,v)}$.



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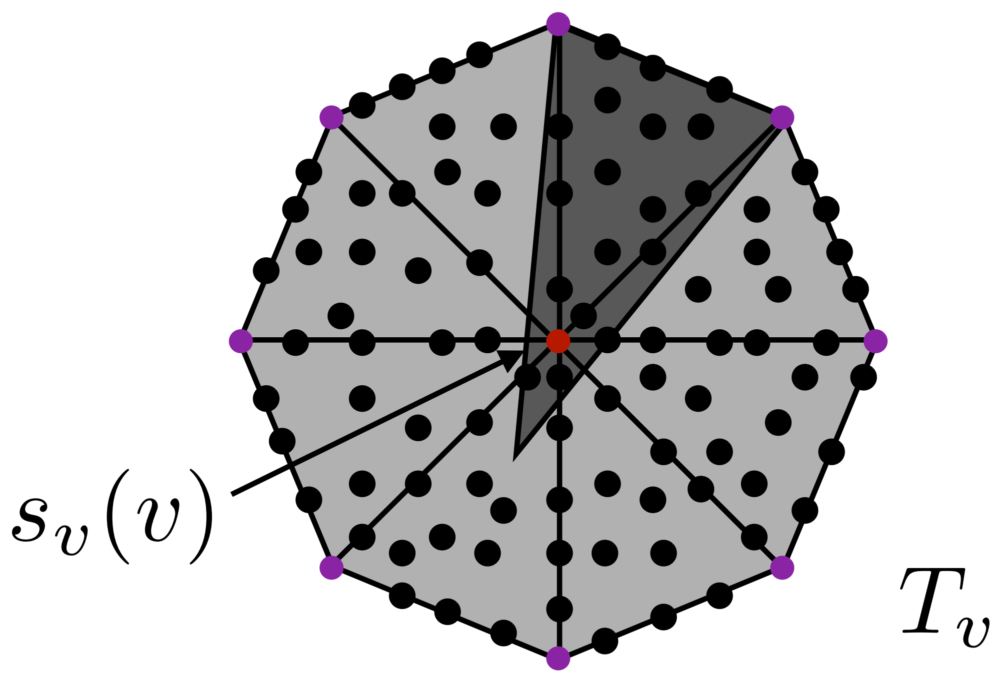

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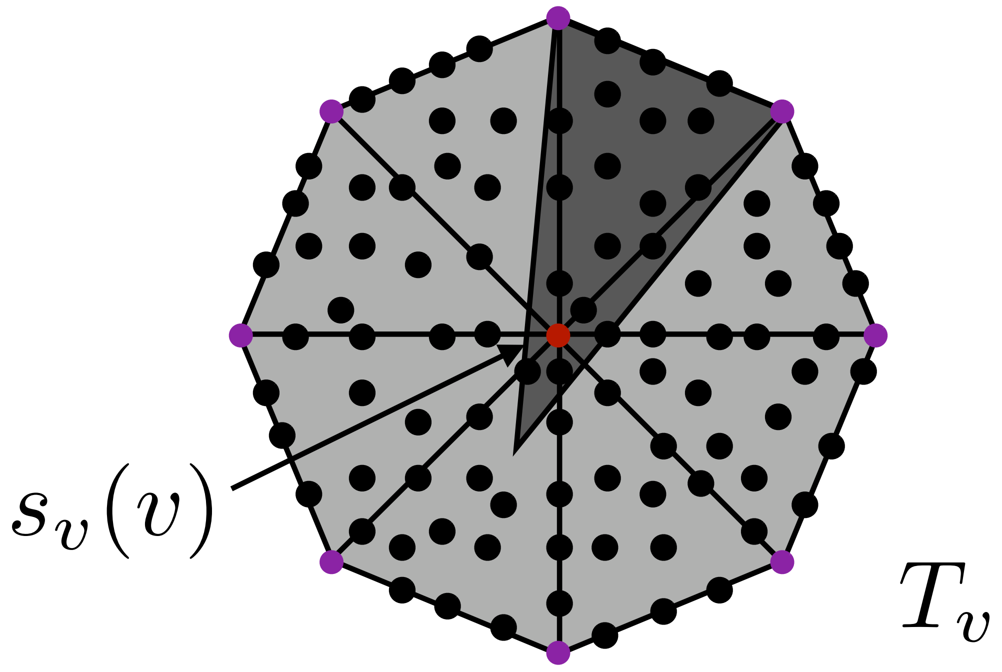

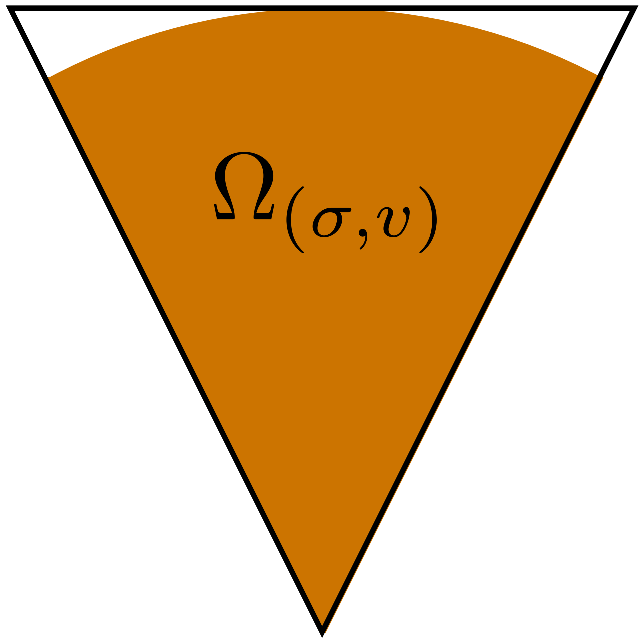

Adaptive Fitting

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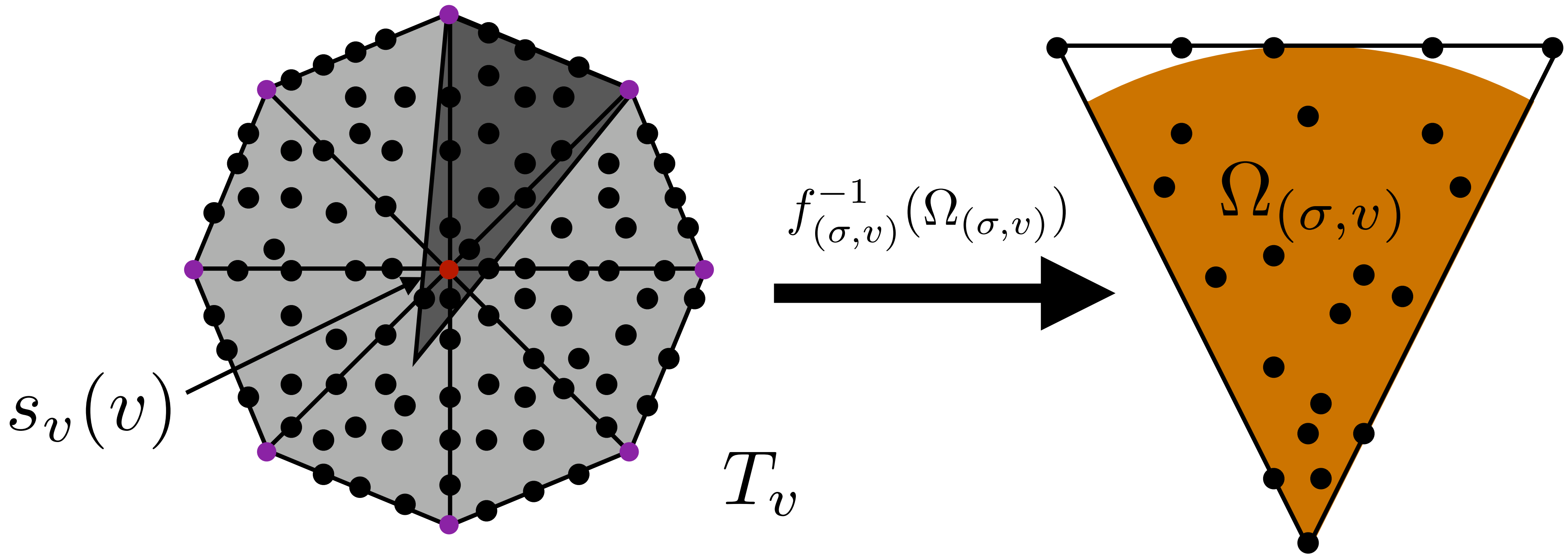

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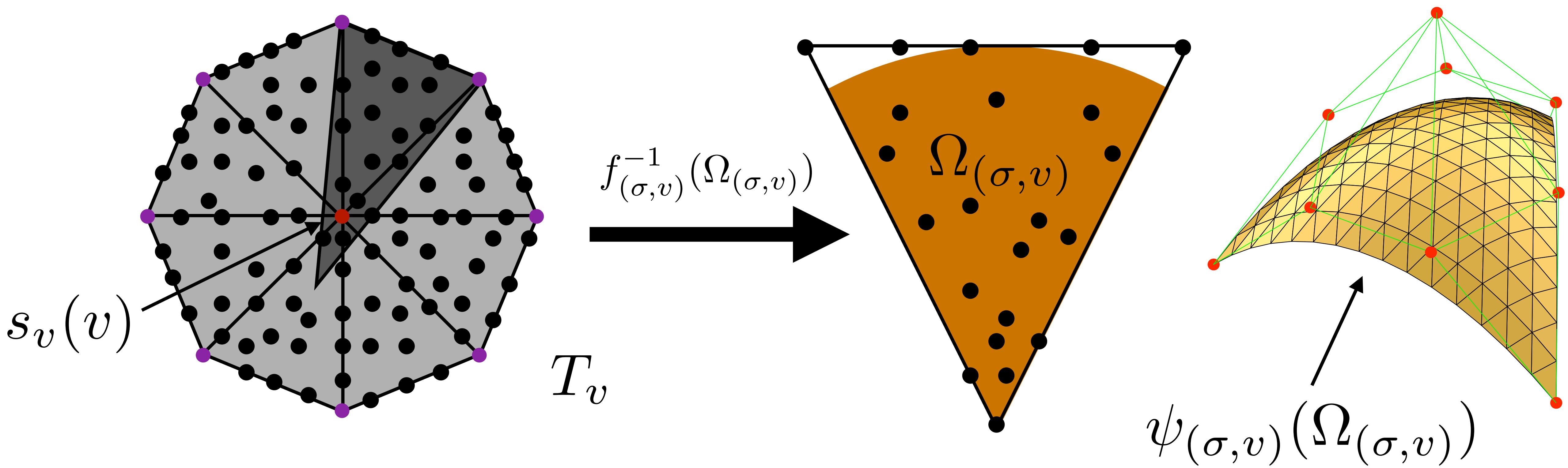

Adaptive Fitting

Create S from $\tilde{S}_T$

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Create S from $\tilde{S}_T$

- Control points of $\psi_{(\sigma,v)}$ are computed by a least squares procedure.

Adaptive Fitting

Create S from $\tilde{S}_T$

- Control points of $\psi_{(\sigma,v)}$ are computed by a least squares procedure.

- But, this time, the sample points are the vertices of S_T that correspond to the points in T_v through Floater's parametrization!

Adaptive Fitting

Create S from $\tilde{S}_T$

Adaptive Fitting

Create S from $\tilde{S}_T$

- For each point p in T_v, we compute the **approximation error**,

$$\|q - \psi_{(\sigma,v)(p)}\|,$$

where q is the vertex of S_T corresponding to p through Floater's parametrization.

Adaptive Fitting

Create S from $\tilde{S}_T$

- For each point p in T_v, we compute the **approximation error**,

$$\left\| q - \psi_{(\sigma,v)(p)} \right\| ,$$

  where q is the vertex of S_T corresponding to p through Floater's parametrization.

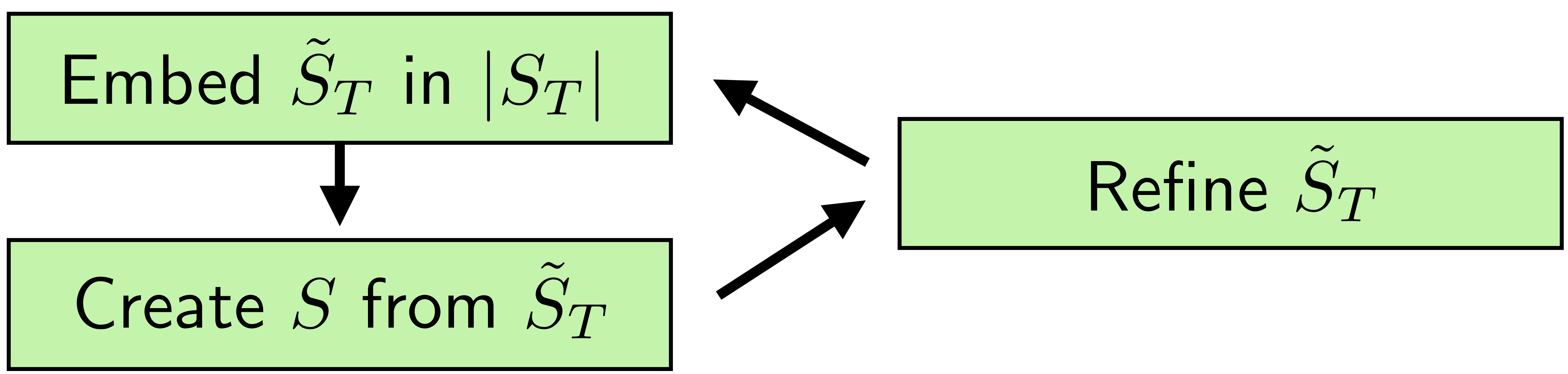
- If the above error is smaller than the given number ϵ, we keep computing $\psi_{(\tau,u)}$, for all pairs $(\tau, u) \in I$. Otherwise, we stop this process and go to the refinement step.

Adaptive Fitting

Adaptive Fitting

Refine $\tilde{S}_T$

- We locally refine $\tilde{S}_T$ using the stellar operations and the 4-8 refinement, and then embed the resulting $\tilde{S}_T$ in $|S_T|$ again.



Discrete Geodesics

Discrete Geodesics

- **Locally Shortest Geodesic**:
  A curve joining two points, A and B, on a polyhedral surface. It is a local minimum of the length functional.

Discrete Geodesics

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  A curve joining two points, A and B, on a polyhedral surface. It is a local minimum of the length functional.

- **Straighest Geodesic**:
  A curve beginning at point A and moving in the direction of the tangent vector. It has zero *discrete geodesic curvature* everywhere.

Discrete Geodesics

Discrete Geodesics

Locally shortest geodesics:

Discrete Geodesics

Locally shortest geodesics:

Exact algorithms:

- Mitchell, Mount, and Papadimitriou (1987)
- Chen and Han (1996)
- Kapoor (1999)
- Surazhsky, Surazhsky, Kirsanov, Gortler, and Hoppe (2005)

Discrete Geodesics

Locally shortest geodesics:

Discrete Geodesics

Locally shortest geodesics:

Approximate algorithms:

- Kimmel and Sethian (1998)
- Martínez, Velho, and Carvalho (2004)
- Surazhsky, Surazhsky, Kirsanov, Gortler, and Hoppe (2005)

Discrete Geodesics

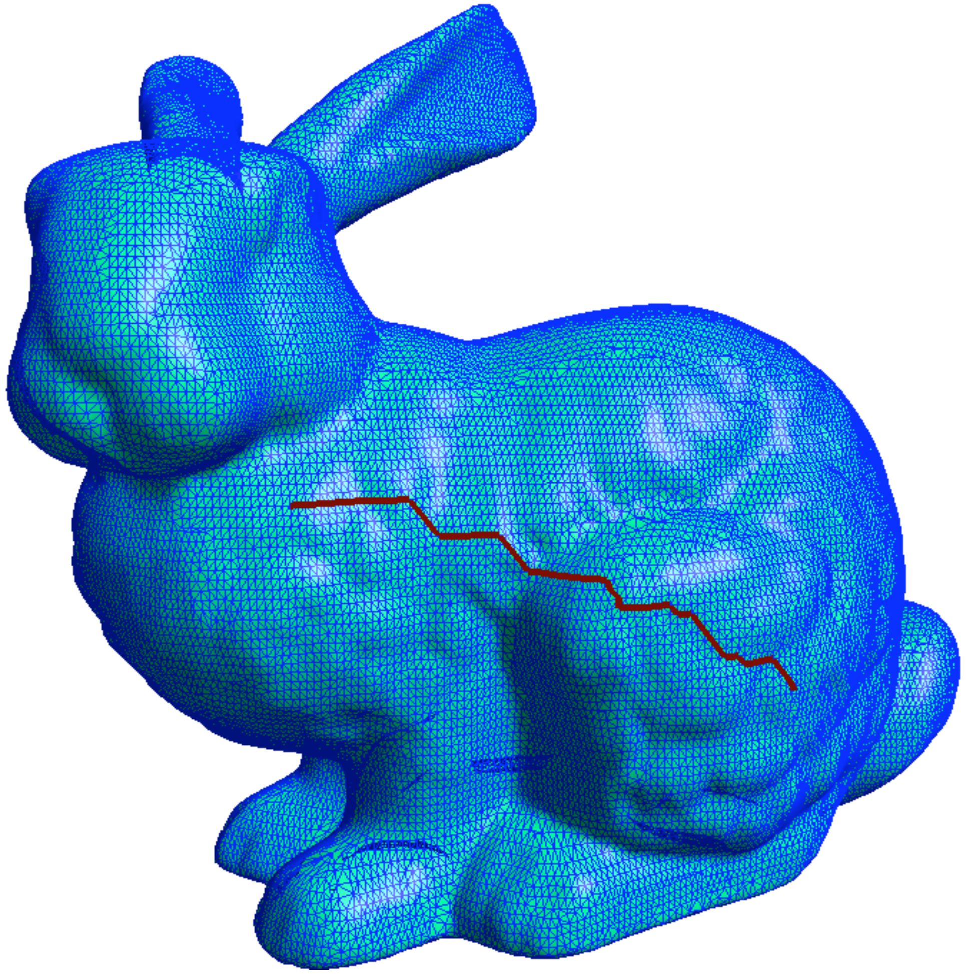
Discrete Geodesics

A Two-Step Algorithm:

Discrete Geodesics

A Two-Step Algorithm:

Step 1:
Find an initial curve joining A and B.



Discrete Geodesics

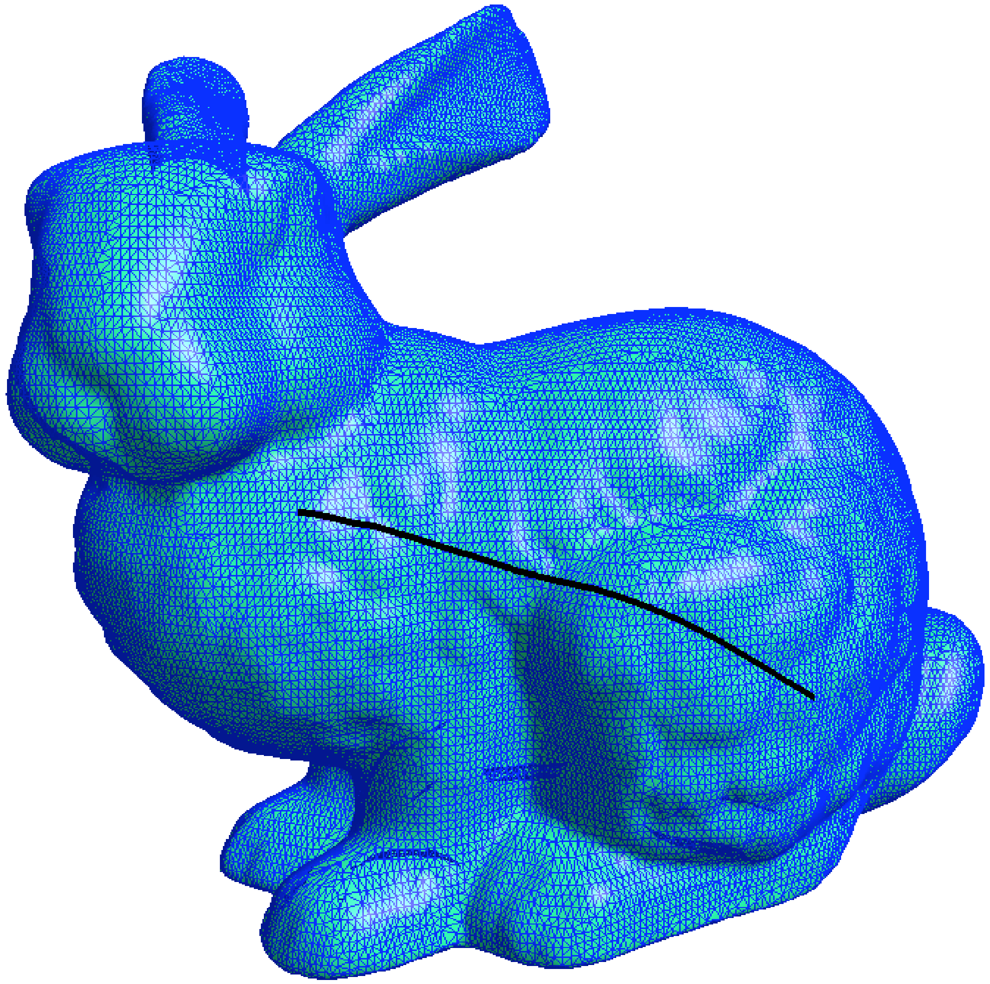
A Two-Step Algorithm:

Discrete Geodesics

A Two-Step Algorithm:

Step 2:

Iteratively modify the position of each curve vertex.



Discrete Geodesics

Discrete Geodesics

Step 1:
Find an initial curve joining A and B.

Discrete Geodesics

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Find an initial curve joining A and B.

- Fast Marching Method

Discrete Geodesics

Step 1:
Find an initial curve joining A and B.

- Fast Marching Method

- Define a distance function at the vertices, $d(v) = \mathrm{dist}(A, V)$, using an approximation of the eikonal equation

$$|\nabla d| = 1 \, .$$

Discrete Geodesics

Step 1:

Find an initial curve joining A and B.

Discrete Geodesics

Step 1:
Find an initial curve joining A and B.

- Back-integrate the differential equation:

$$\begin{cases} \dfrac{d\Gamma_0}{ds}(s) & = & -\nabla d(\Gamma_0(s)) \\ \\ \Gamma_0(0) & = & B\,. \end{cases}$$

Discrete Geodesics

Discrete Geodesics

Step 2:

Iteratively modify the position of each curve vertex.

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  - a geodesic should be a line segment in the interior of a face;
  - the curve will be a polygonal with nodes belonging to the edges of the mesh;
  - the algorithm will correct the position of the curve nodes;
  - distinct behavior for "edge nodes" and "vertex nodes".

Discrete Geodesics

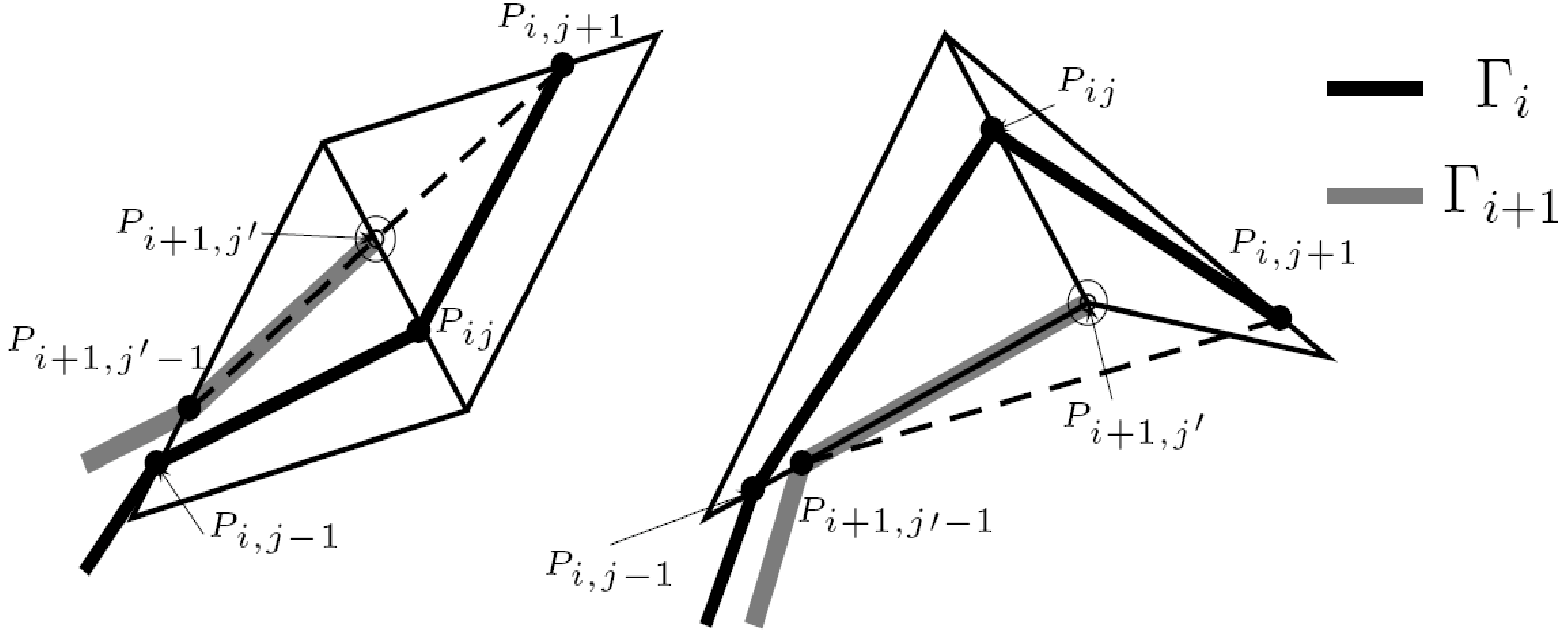
Step 2:

Iteratively modify the position of each curve vertex.

Discrete Geodesics

Step 2:
Iteratively modify the position of each curve vertex.

Edges nodes:



Discrete Geodesics

Step 2:
Iteratively modify the position of each curve vertex.

Discrete Geodesics

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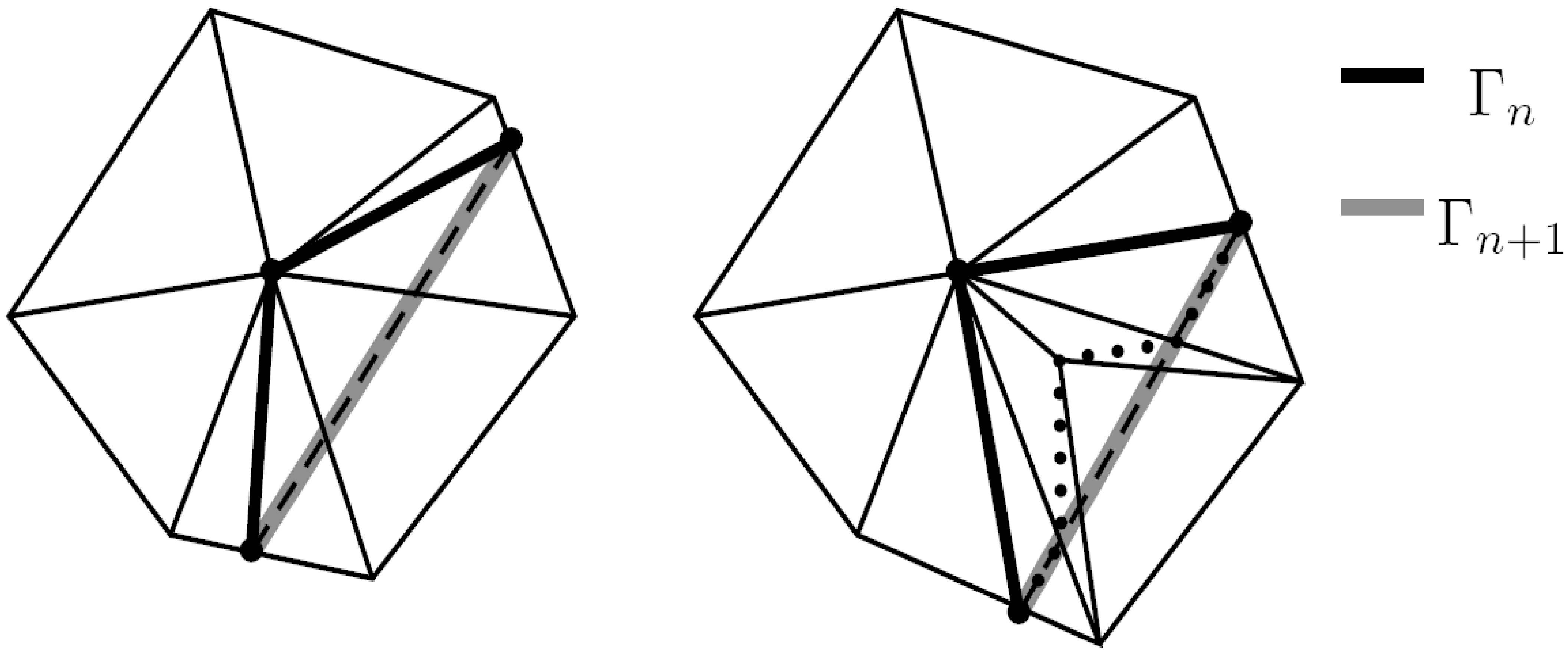
Vertex nodes:



Discrete Geodesics

Discrete Geodesics

Examples:

Discrete Geodesics

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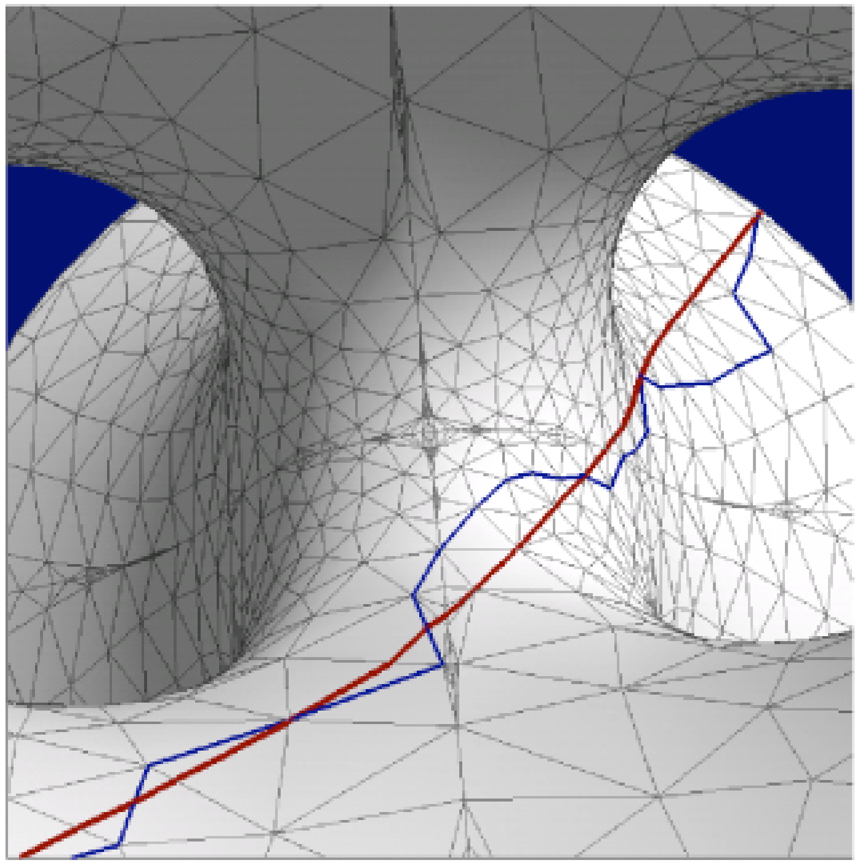

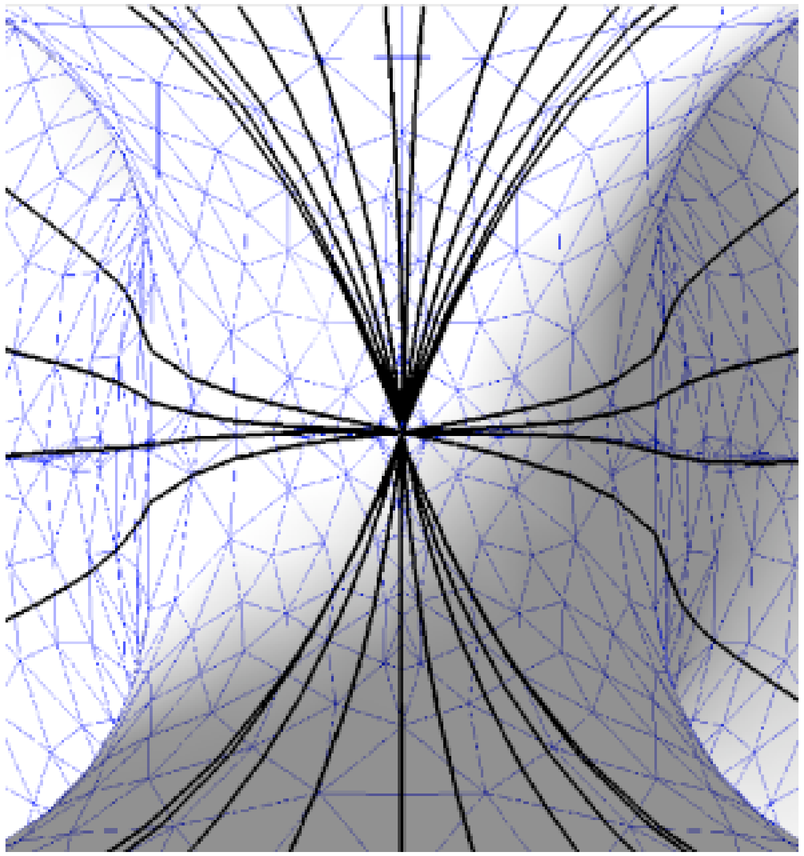

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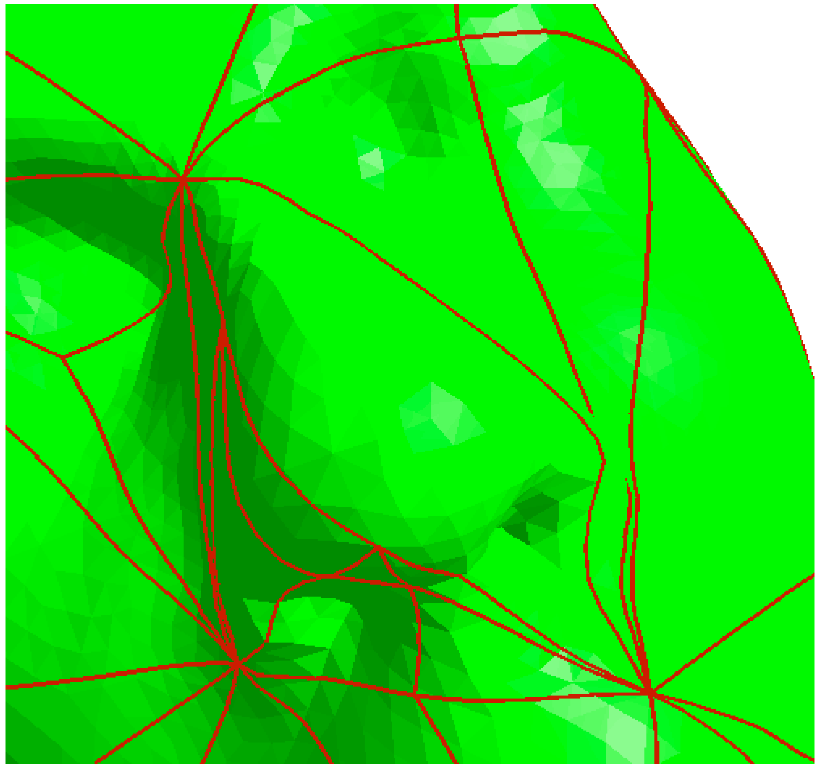
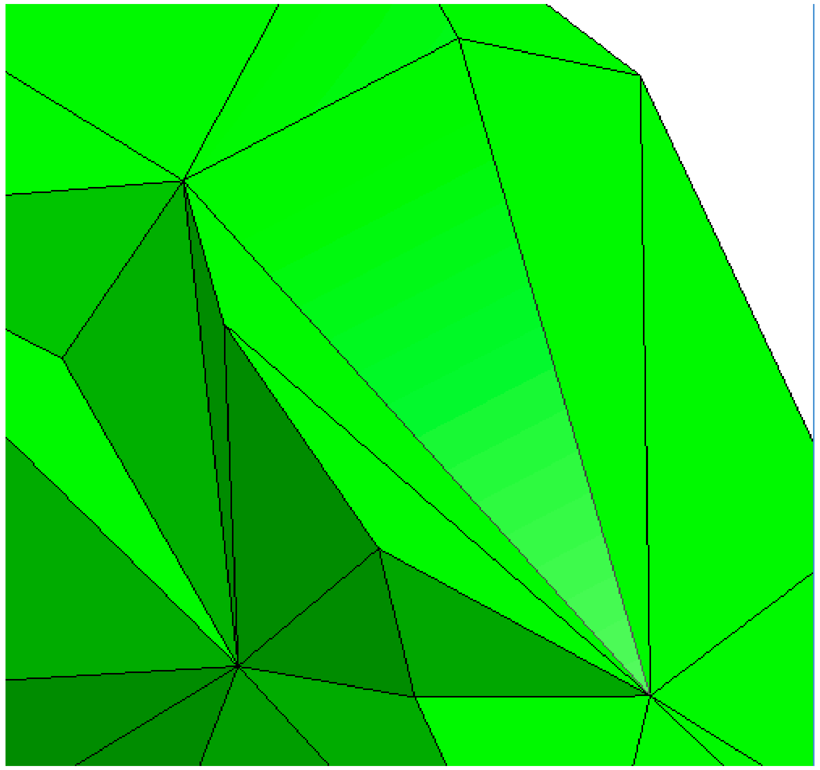
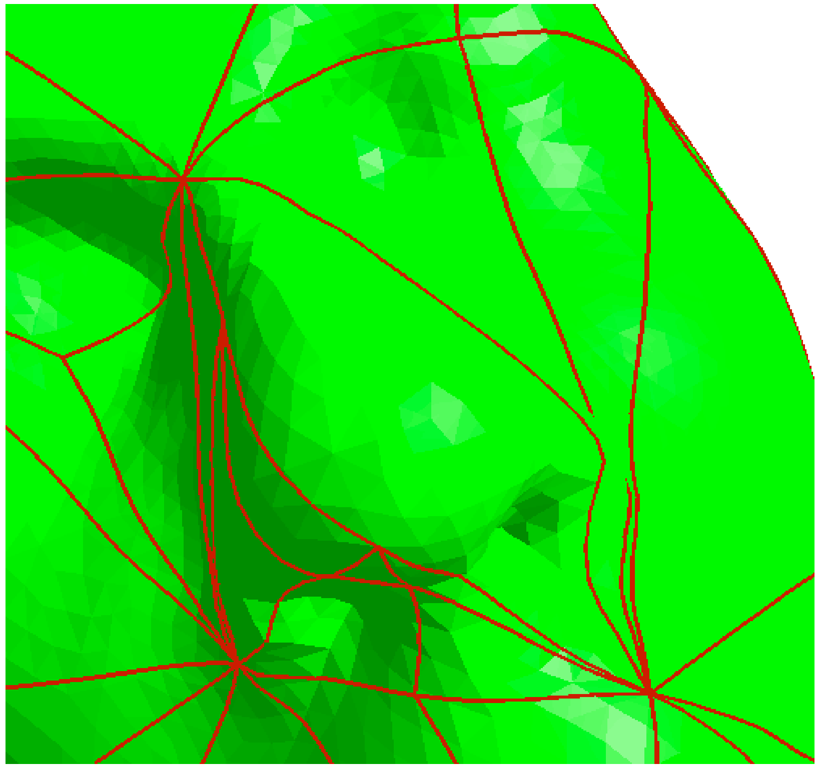


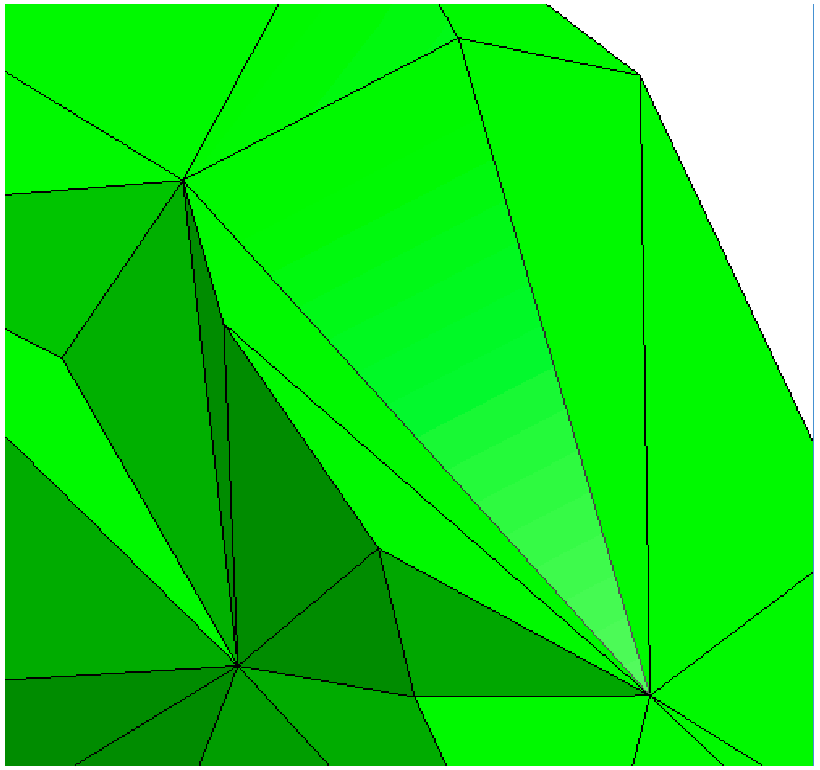


Discrete Geodesics

Adaptive Fitting:

When defining geodesic triangles, we can violate the manifold property of the geodesic mesh, as illustrated by the figure below:





Discrete Geodesics

Discrete Geodesics

Solution:

Compute the geodesic curve AB as the geodesic resulting from using the concatenation of geodesics AC and CB as initial approximation:

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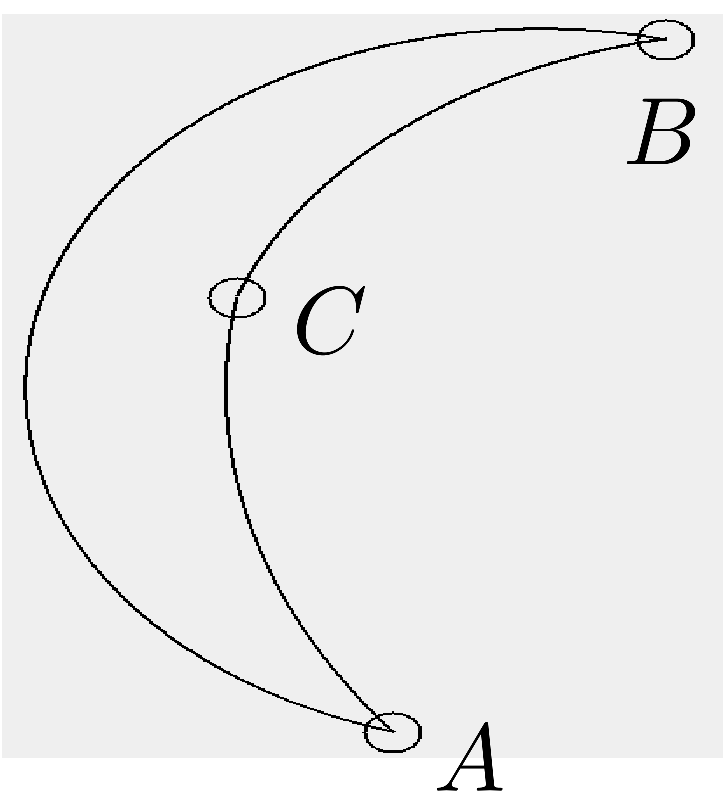

Discrete Geodesics

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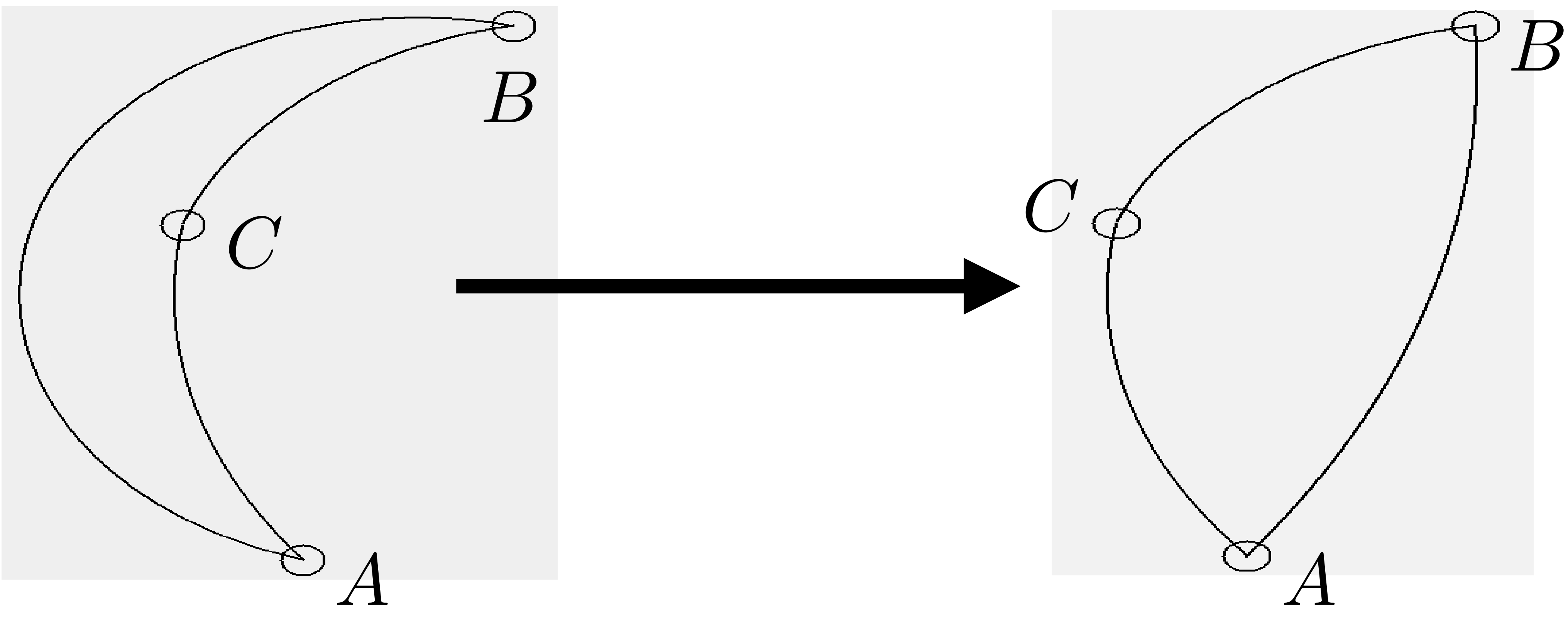

Conclusions

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- This is the "real deal" when it comes to comparisons between smooth surfaces and very dense polygonal meshes.
- Implementation of the adaptive fitting is still under development.
- More specifically, the refinement step has not been completed.

Applications of Manifolds and Research Challenges

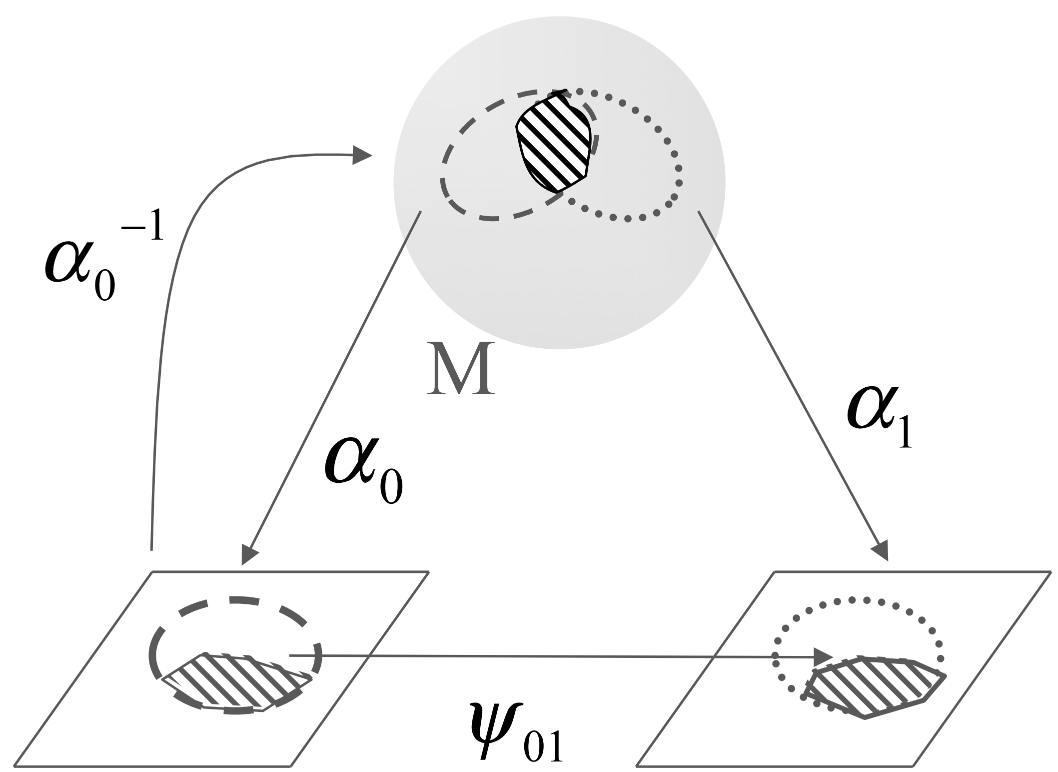
Luiz Velho
IMPA

Outline

- Concepts
- Illumination
- Appearance
- Simulation
- Faces
- Manifold Learning
- Wrap-up

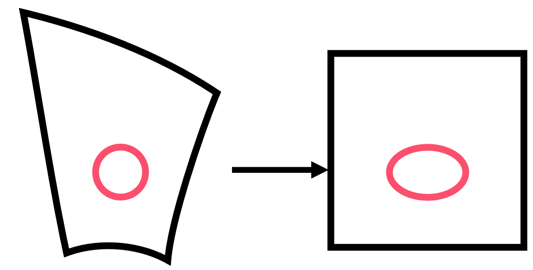
Manifolds & Parametrization

- Two Points of View
  - Functions on surfaces
  - Functions defining surfaces

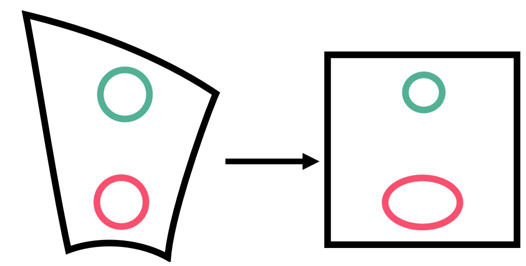


Desirable Properties

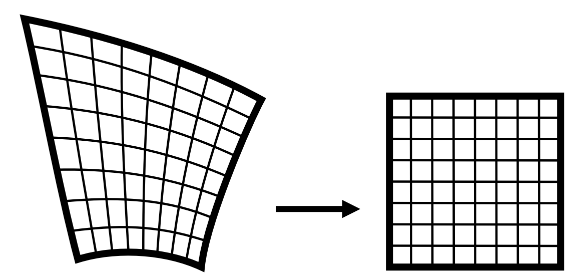
- Minimal Distortion
  - Angle



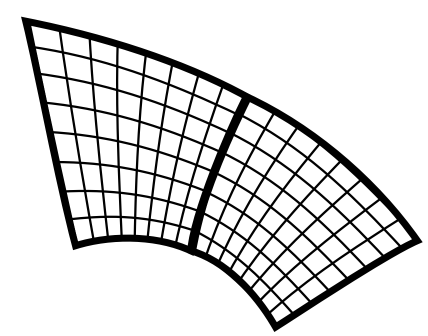
  - Area



- Smoothness
  - Differentiability



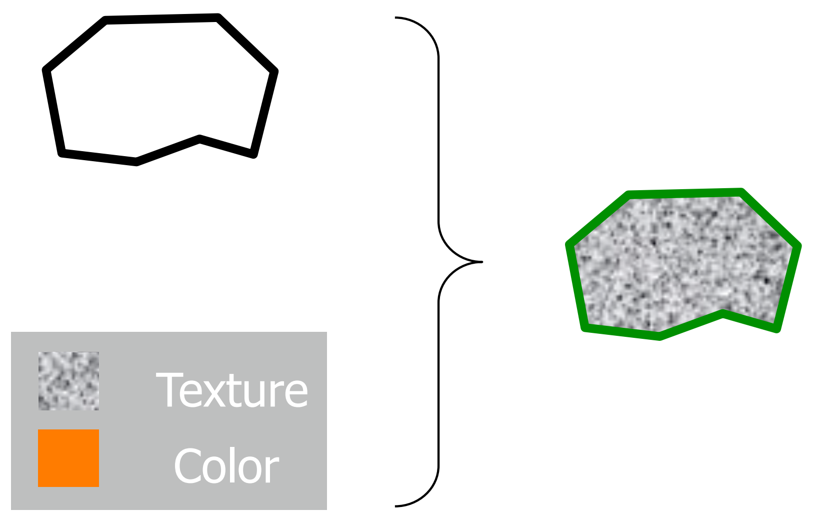
  - Continuity



Graphical Objects

- Shape U
  - Topology (*domain*)
    - Abstract Manifold
  - Geometry (*function*)
    - Embedding
- Attributes f
  - Functions (*co-domain*)

$$O = (U, f)$$



G.O. Manifold Setting

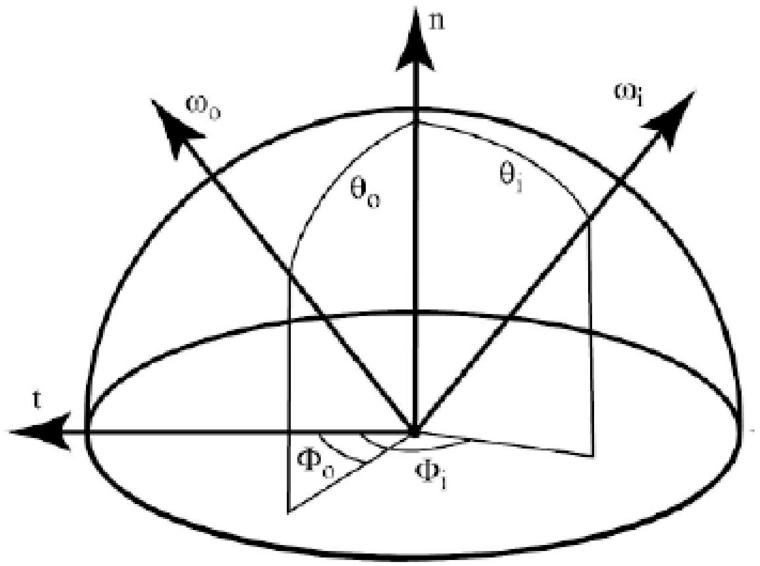
- Canonical Surfaces
  - Fixed Shape (defined *apriori*)
  - Variable Functions (complex)
    - *ex: Sphere*
- Arbitrary Surfaces
  - Complex Shape
  - Computation on Surfaces (attributes)
  - Building / Transforming (shape)
    - *ex: Triangle Meshes*

Applications

- Illumination
  - Canonical Manifold + Functions
- Appearance and Simulation
  - Pseudo-Manifold + Attributes
- Faces
  - Manifold + Geometric Deformation
- Surface Reconstruction
  - Pseudo-Manifold / Topology Estimation

Illumination

- Functions on the Sphere
  - Light Fields / BRDFs
- Applications
  - Capture / Synthesis
- Construction [Grimm 2002]



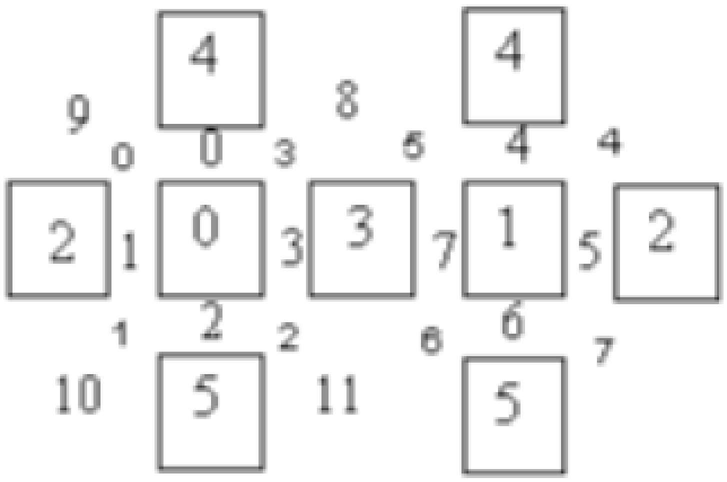
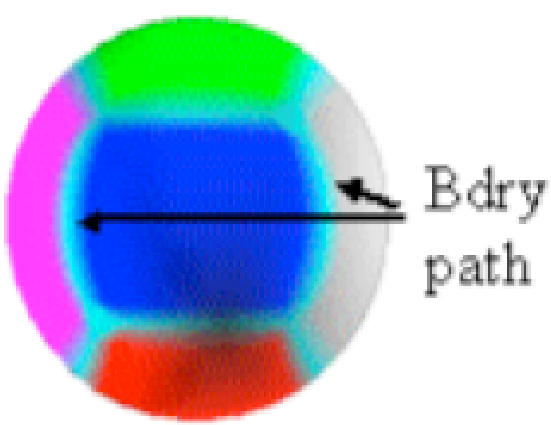
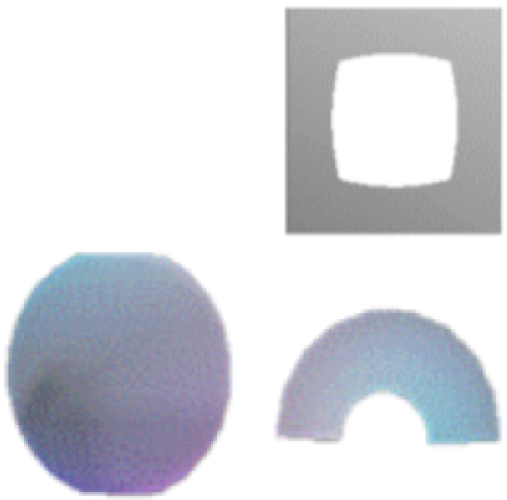


Chart (squares), edge, and



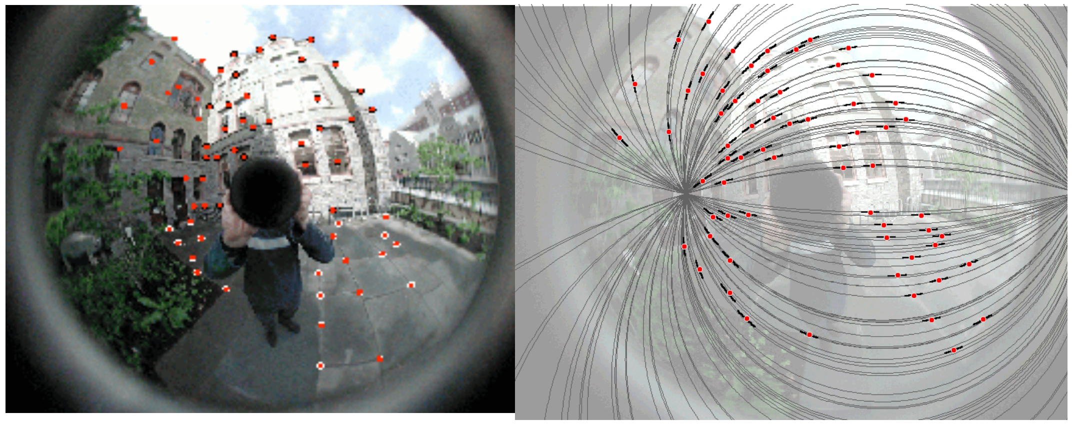
Bottom cap



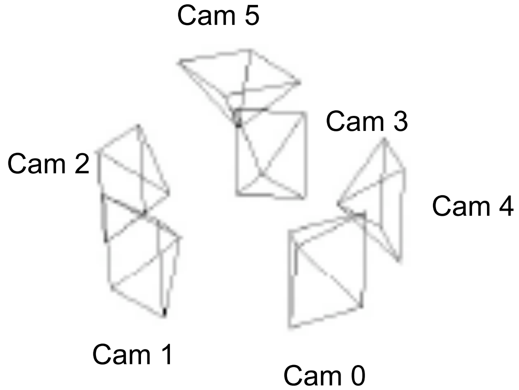
A single chart on the sphere

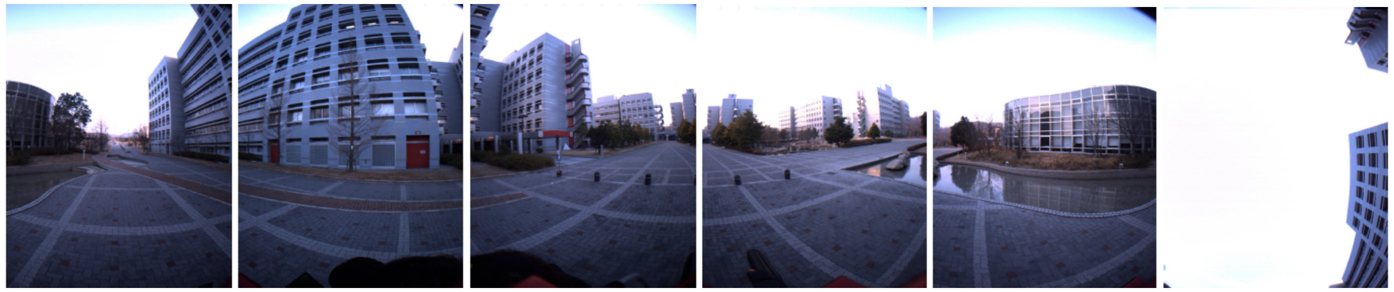
Omnidirectional Images

- Panoramic Cameras
  - Processing



- Multi-Camera Assembly
  - Stitching / Blending

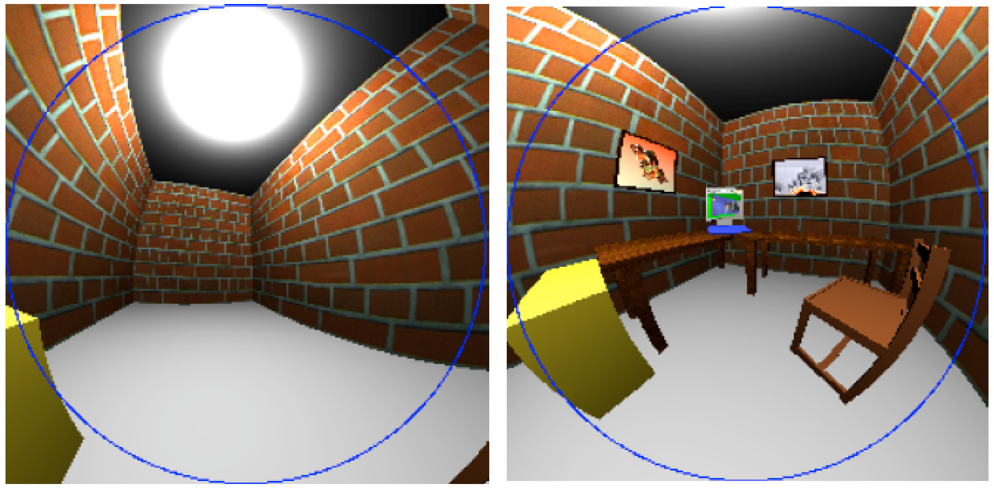


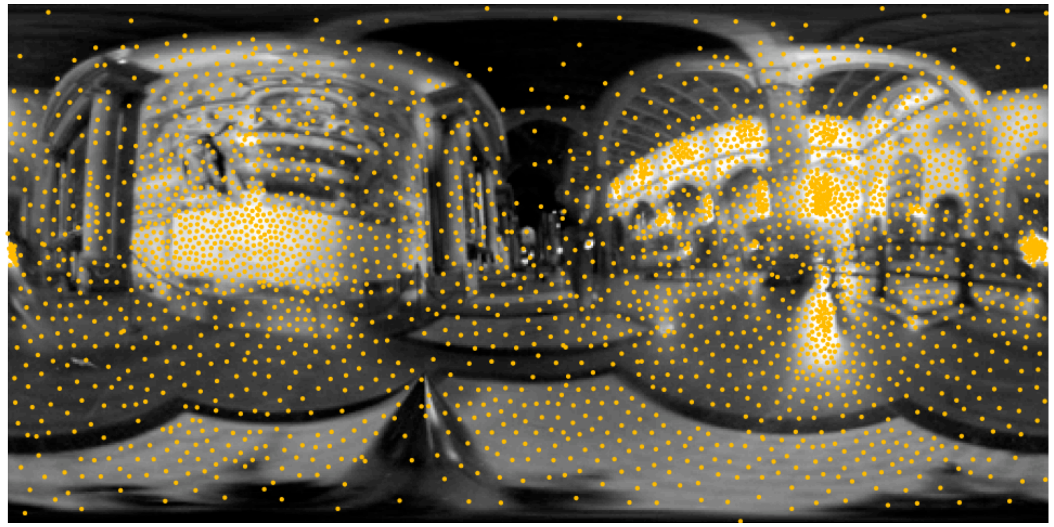


Cam 0 Cam 1 Cam 2 Cam 3 Cam 4 Cam 5

Illumination Maps

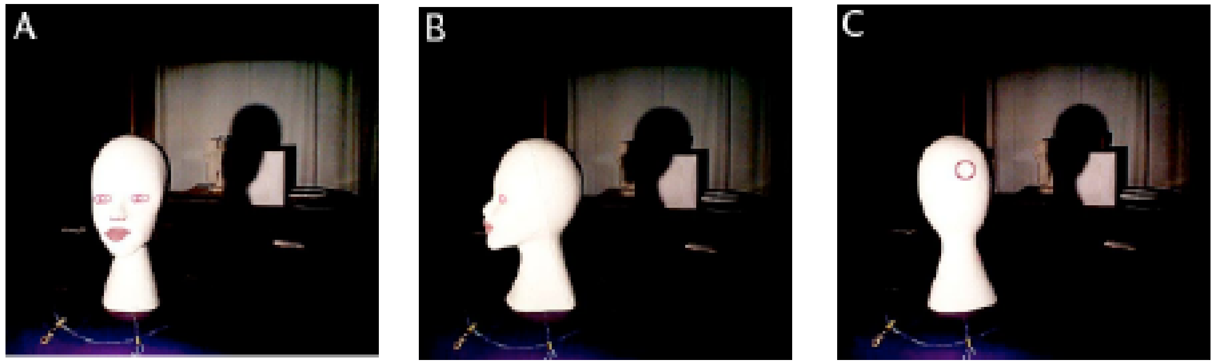
- Environment Maps
  - Area Sampling
- Light Maps
  - Stratification

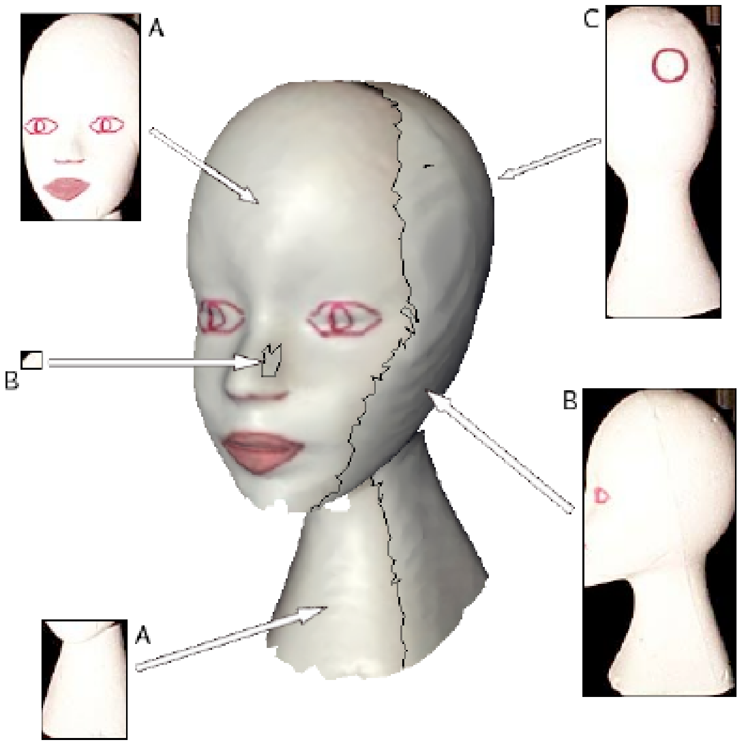




Surface Properties

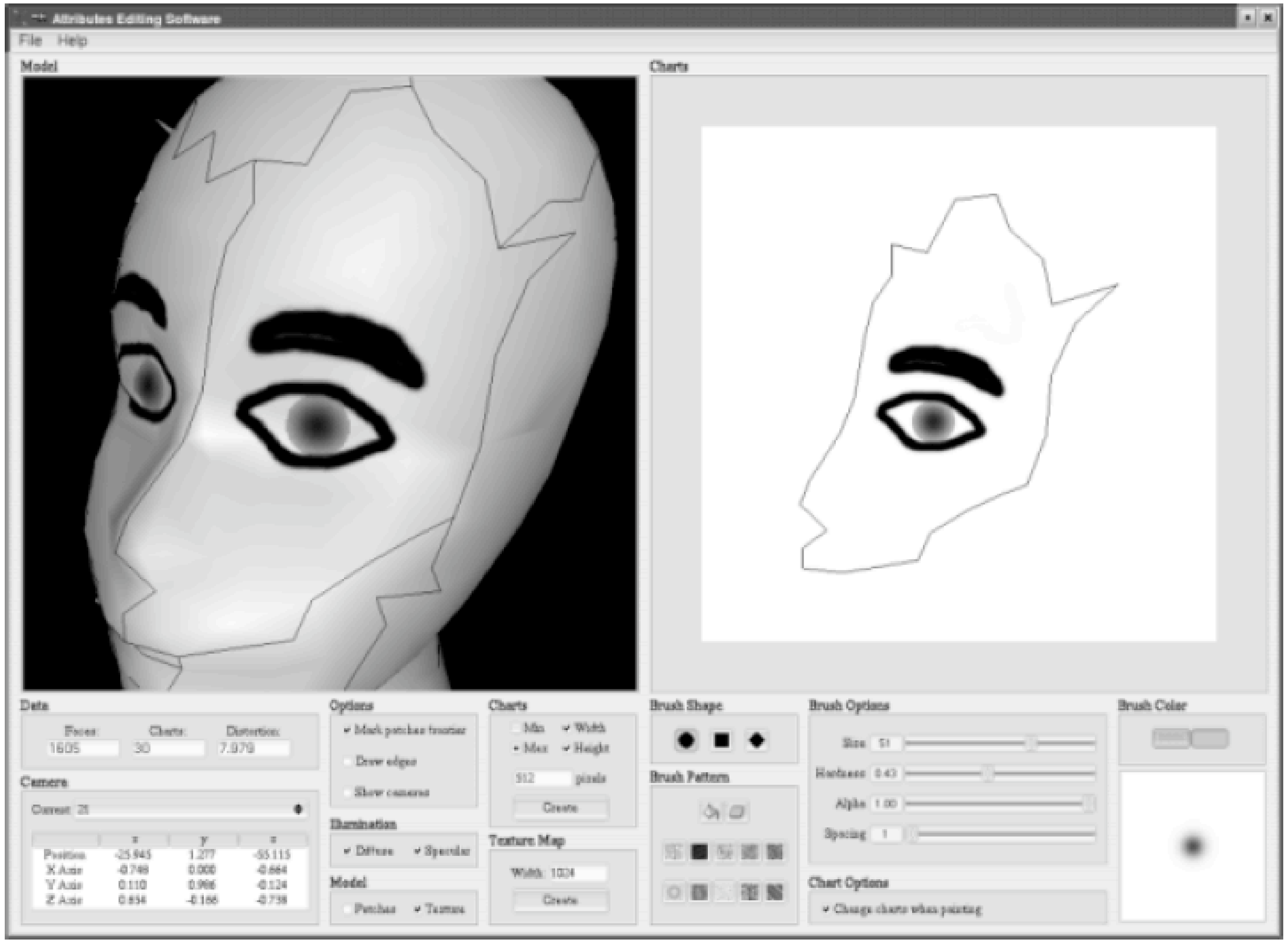
- Texture Atlas
  - Albedo
  - Normal Field
- Building from Images
  - Projective Map





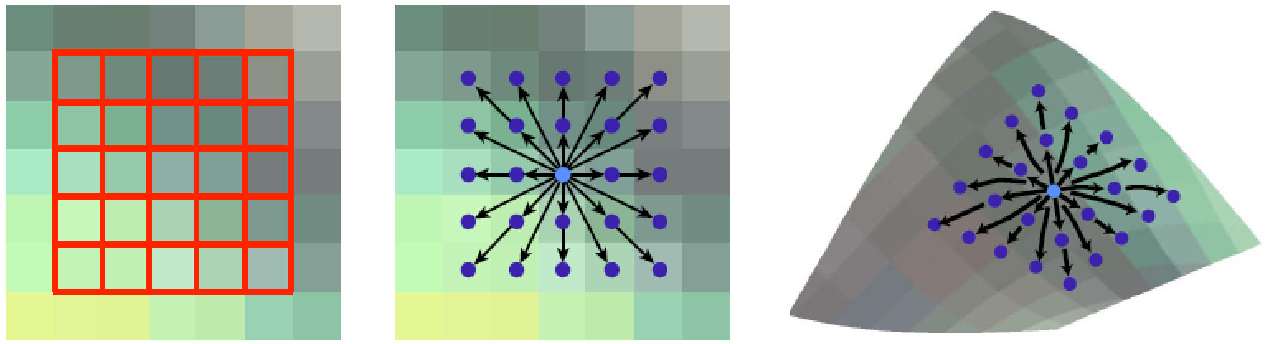
Painting

- Color
- Normals



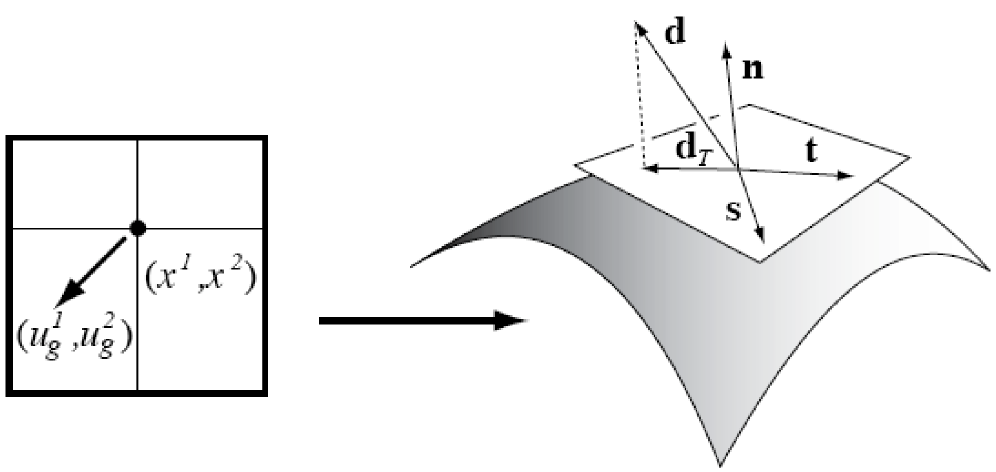
Texture Synthesis

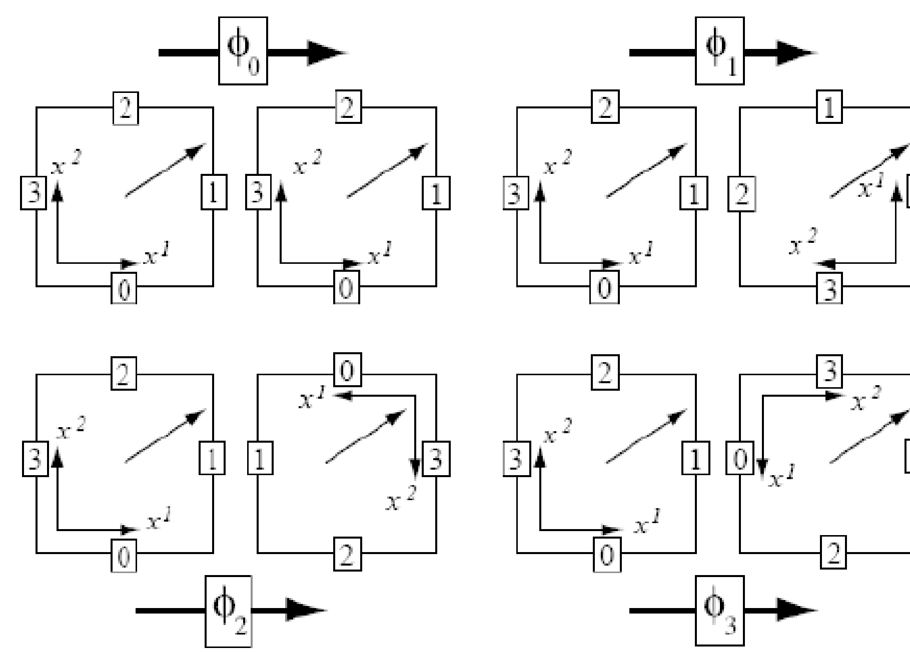
- Stationary / Quasi Stationary



Simulation

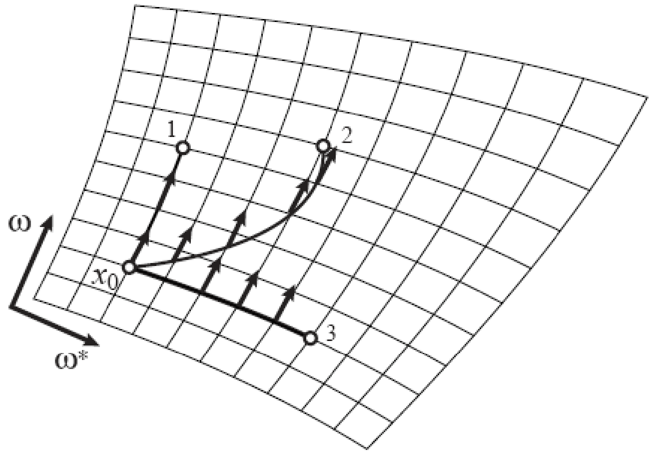
- Solving Equations on Manifolds
  - Surface Points
  - Local Neighborhoods

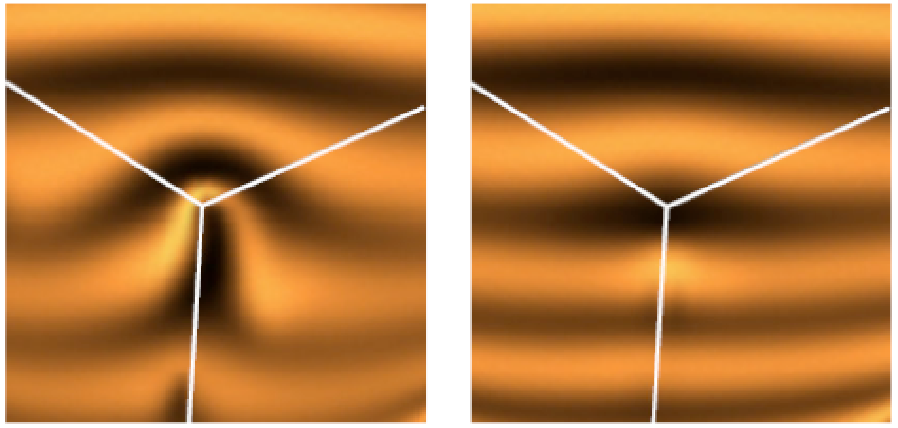


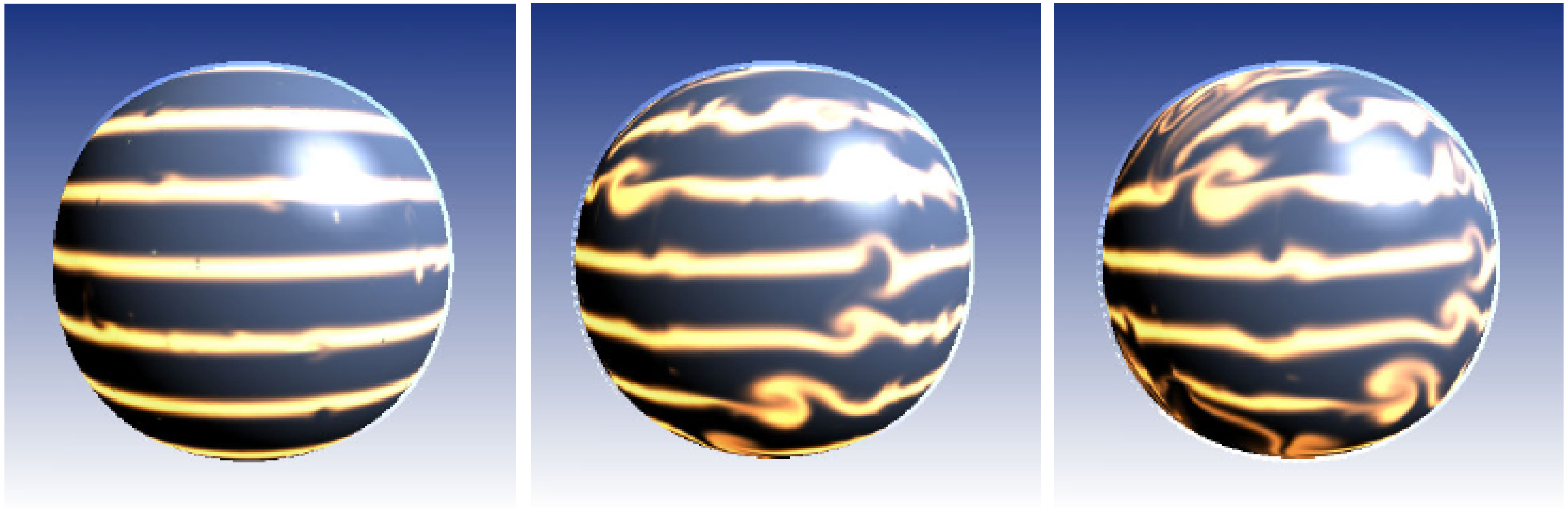


Fluids

- Vector Fields on Surfaces

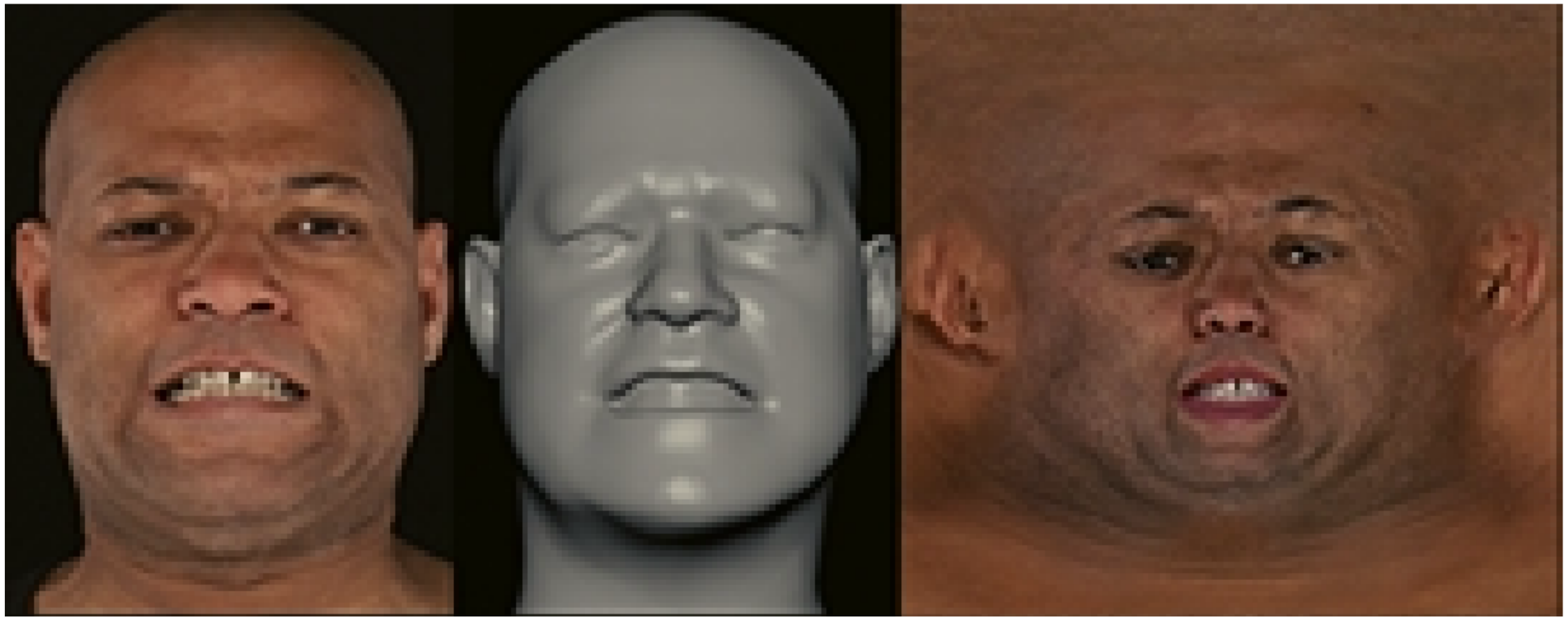






Faces

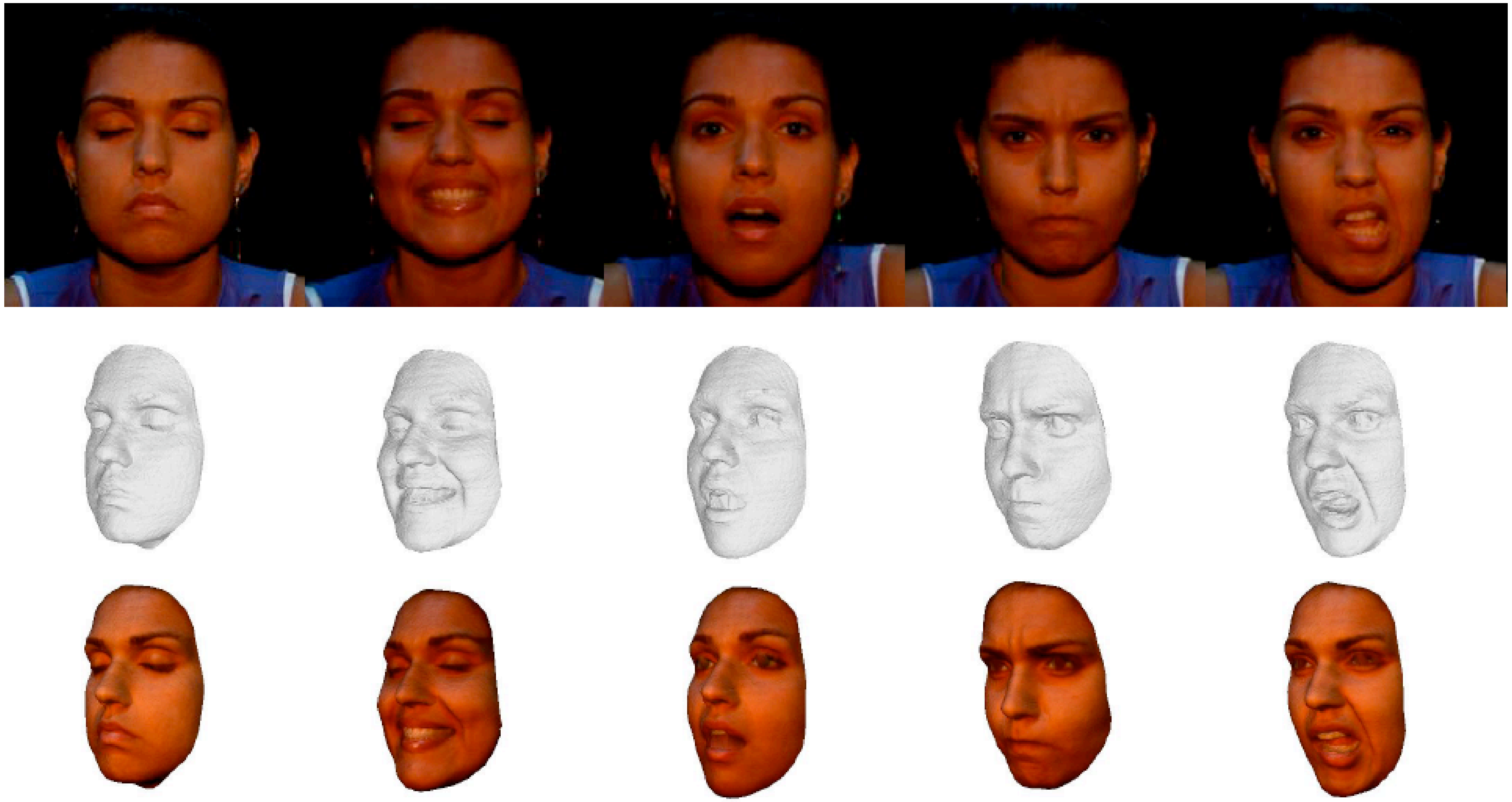
- Geometry + Appearance



[G. Borshukov et al SIGGRAPH 2003]

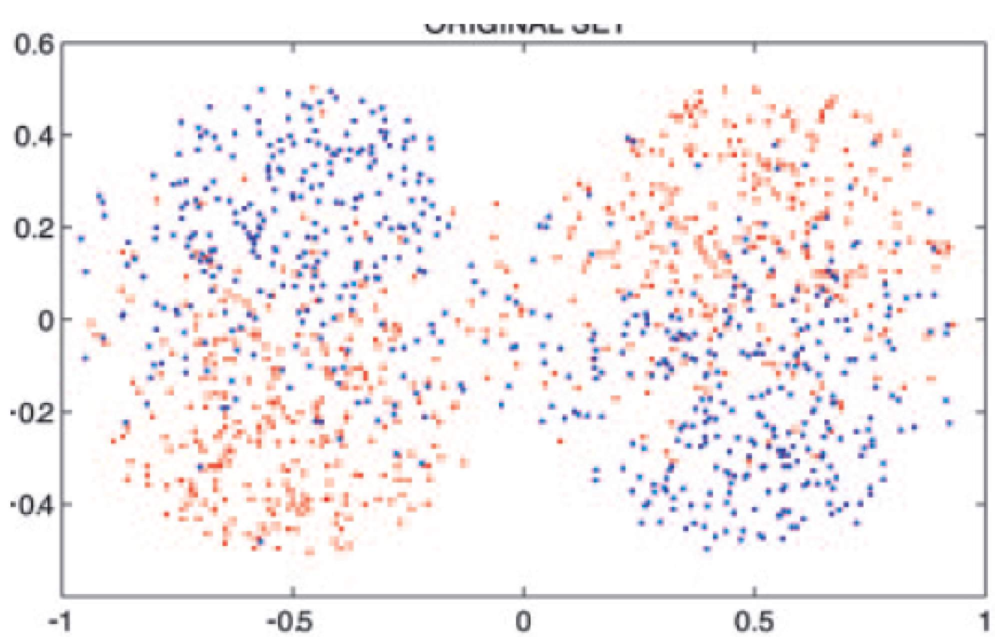
Facial Expressions

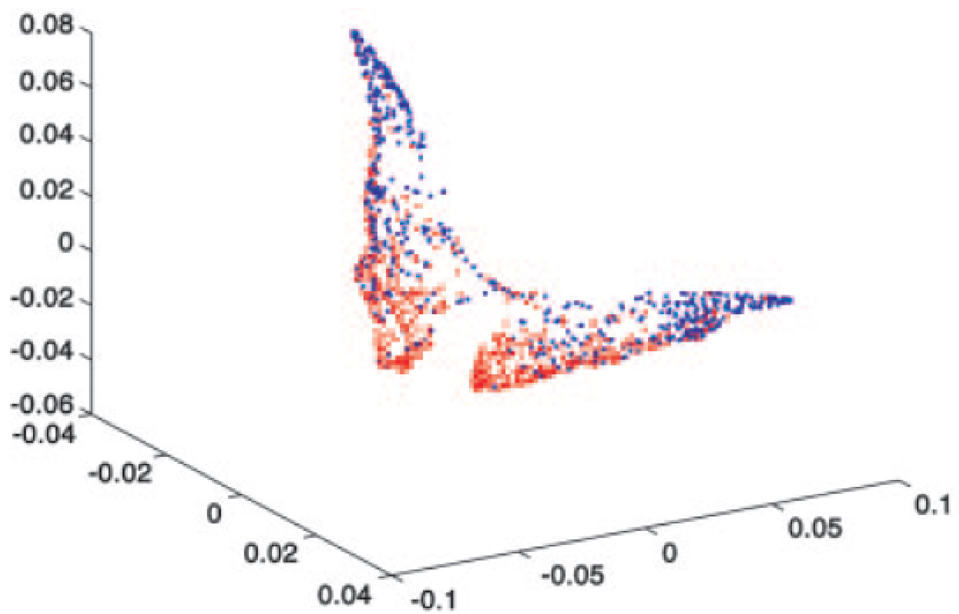
- Deformations



Manifold Learning

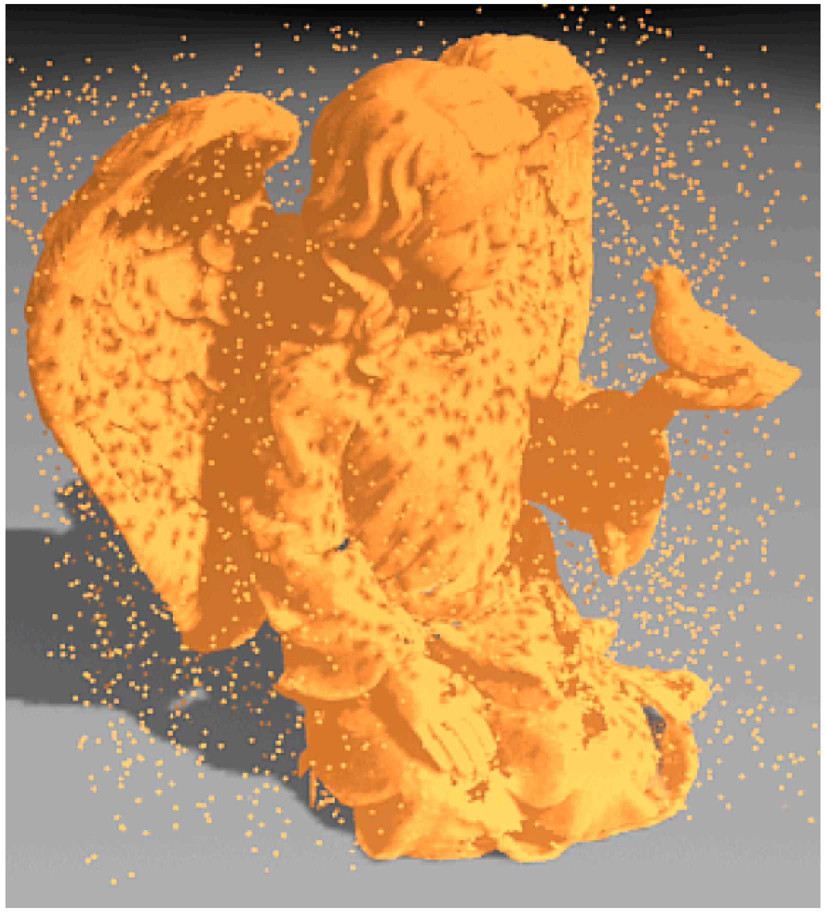
- Estimate from Data Samples
  - Topology
  - Geometry

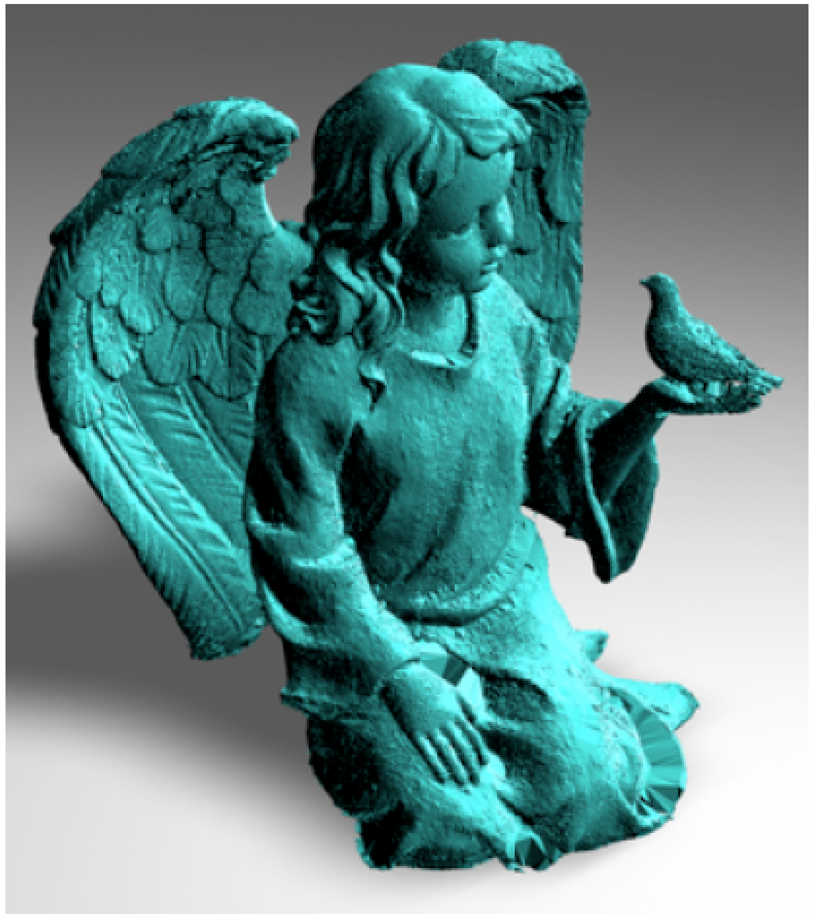




Surfaces

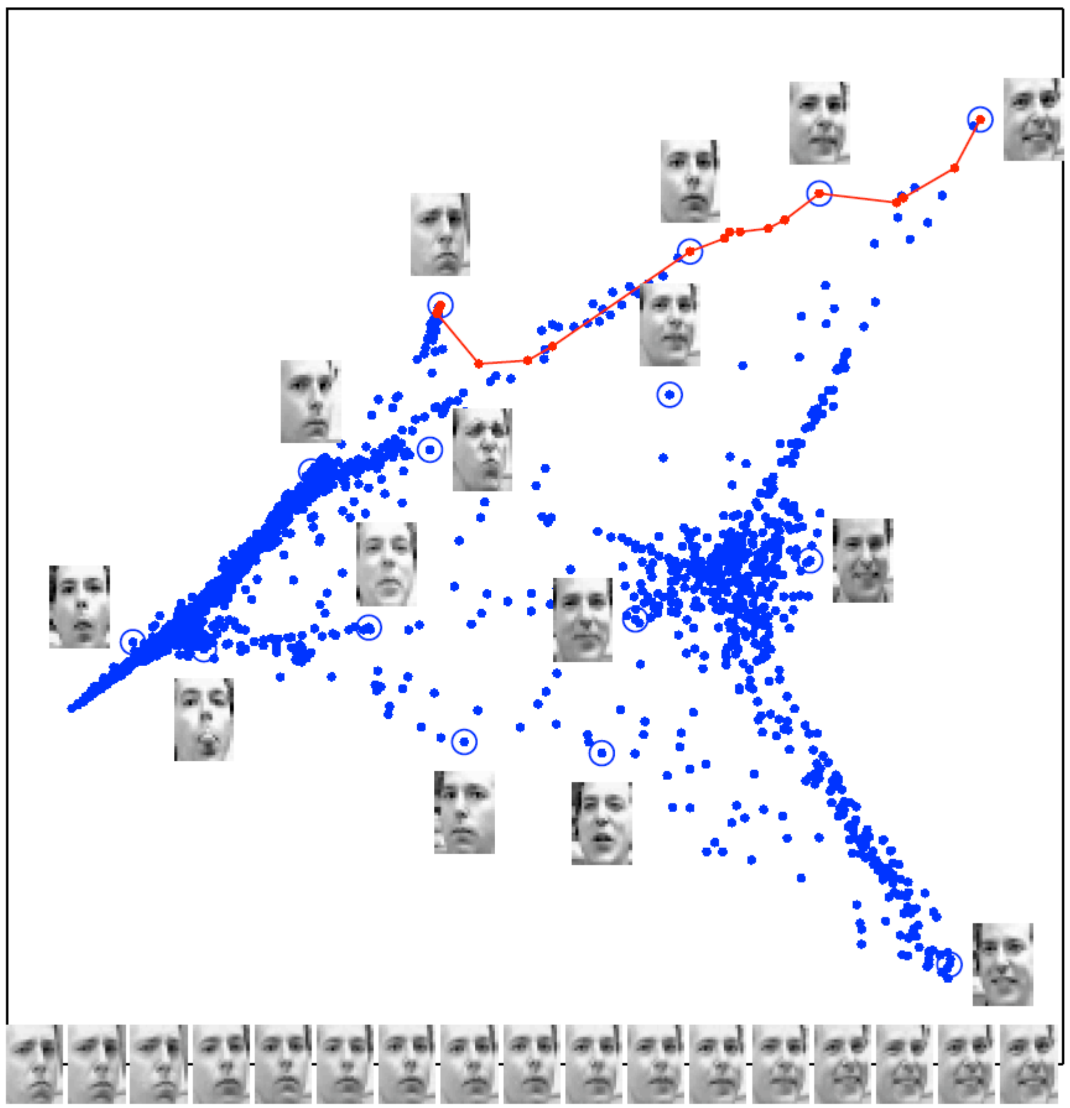
- Point Sets

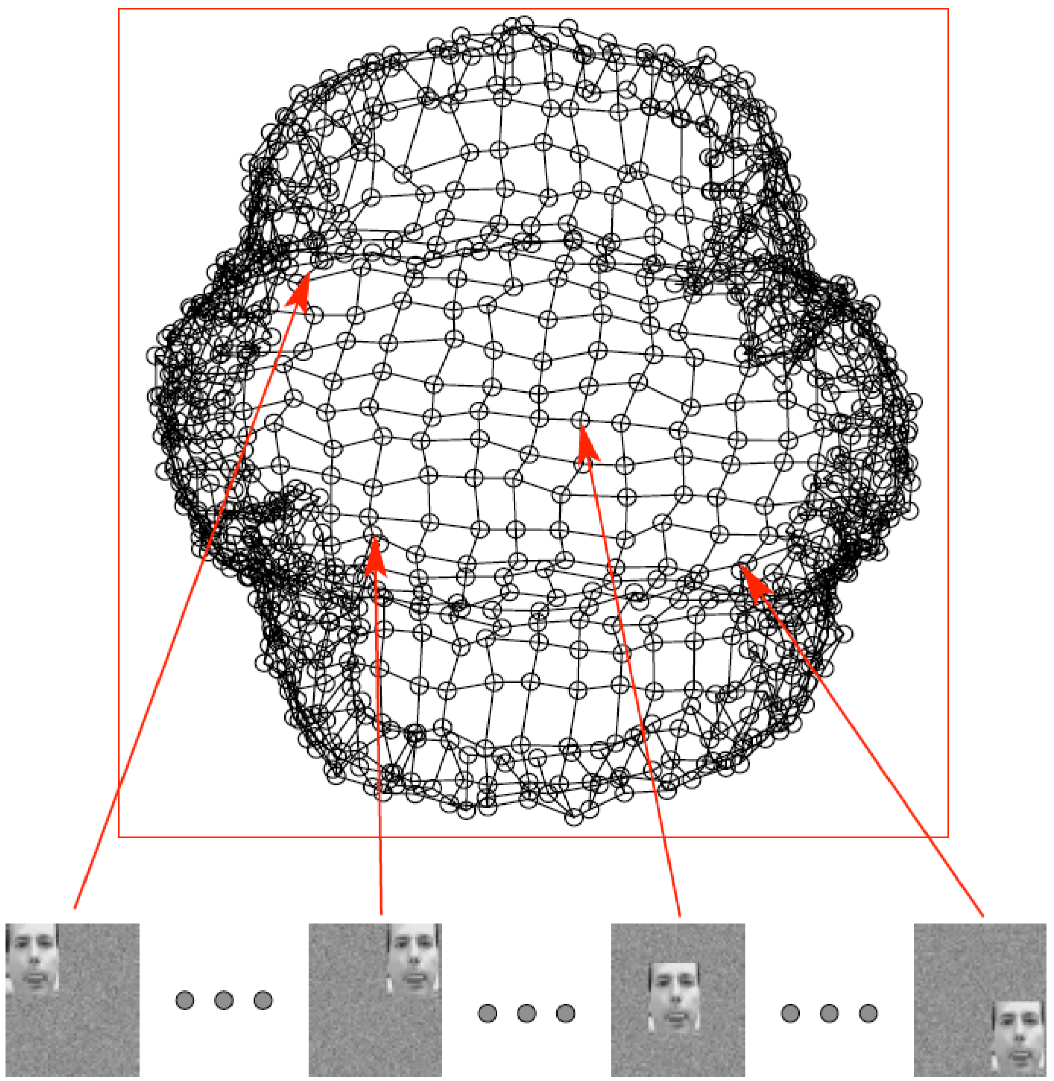




N-Dimensional Case

- ex: Facial Expressions





Challenges

- Multi-Resolution
  - Hierarchical Atlas
  - Dynamic Setting
- API
  - Intuitive
  - General

Questions ?