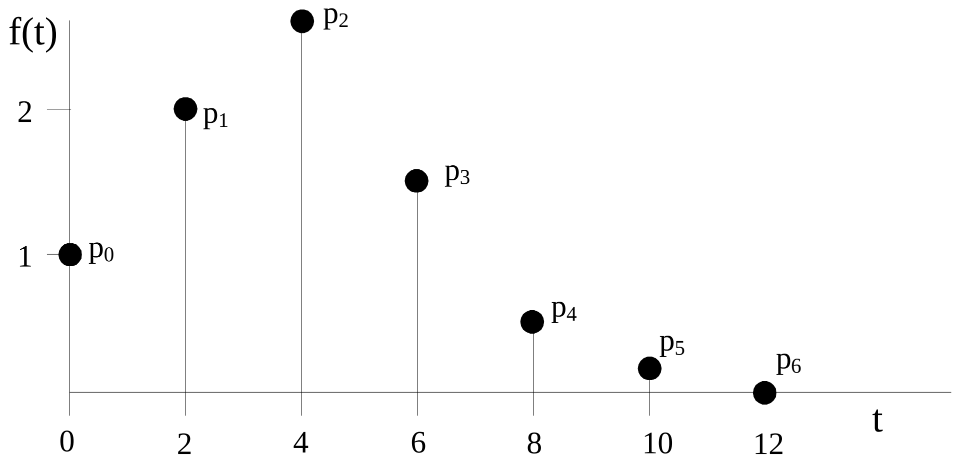
Bspline Construction Summary



Objective of Bspline interpolation:

Given the points (t_0,p_0), (t_1,p_1), (t_2,p_2), (t_3,p_3), (t_4,p_4), (t_5,p_5), (t_6,p_6), … (t_m,p_m), find the coefficients, c_j , j = 0 to m+n-1 of the Bspline basis functions, $N_j^n(t)$, such that the sum,

$$f(t) = \sum_{j=0}^{m+n-1} c_j N_j^n(t)$$

produces an interpolating spline of degree n with values, $f(t) = p_j$ when $t = t_i$. i = 0 to m.
m also represents the number of curve segments between t_0 between t_m , which in the example shown above is 6.

Determining the coefficients, c_j , of an interpolating Bspline:

Step 1. Assemble the knot vector of the form $\boldsymbol{\lambda} = [\lambda_0 \quad \lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_{m+2n}]$ containing the values of t where the polynomial curve segments of degree n are to be joined.

Note: Since n+1 Bspline basis functions $N_j^n(t)$ of degree n are required to be computed over each interval $t \in [t_j, t_{j+1}]$, n additional knot points must be defined to the left of t_0 and to the right of t_m in order to enable $f(t)$ to be computed for the intervals $t \in [t_0, t_1]$ and $t \in [t_{m-1}, t_m]$.

Additional left side knots: $t_{-k} = t_0 - k(t_1 - t_0)$ k = 1 to n
Additional right side knots: $t_{m+k} = t_m + k(t_m - t_{m-1})$ k = 1 to n

When n = 3, this results in a knot vector of the form

$$\boldsymbol{\lambda} = [t_{-3} \quad t_{-2} \quad t_{-1} \,\vdots\, t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m \,\vdots\, t_{m+1} \quad t_{m+2} \quad t_{m+3}]$$
$$= [\lambda_0 \quad \lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_{m+2n}]$$

When n = 1, the knot vector would be of the form

$$\boldsymbol{\lambda} = [t_{-1} \,\vdots\, t_0 \quad t_1 \quad t_2 \quad \ldots \quad t_m \,\vdots\, t_{m+1}]$$
$$= [\lambda_0 \quad \lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_{m+2n}]$$

For example, given the seven points (t_i, p_i), i=0 to 6 shown on the figure of the first page,

For the cubic case (n=3) the knot vector, $\boldsymbol{\lambda}$, would take the form,

$$\boldsymbol{\lambda} = [-6 \quad -4 \quad -2 \,\vdots\, 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \,\vdots\, 14 \quad 16 \quad 18]$$
$$= [\lambda_0 \quad \lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_{12}]$$

For the linear case (n=1) the knot vector, $\boldsymbol{\lambda}$, would take the form,

$$\boldsymbol{\lambda} = [-2 \,\vdots\, 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \,\vdots\, 14]$$
$$= [\lambda_0 \quad \lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_8]$$

<u>Step 2</u>. For each i (i = 0 to m), compute the set of $N_j^n(t_i)$ that are nonzero over the interval $t_i \le t < t_{i+1}$ using the recurrence equation,

$$N_j^n(t) = \frac{(t - \lambda_j)}{(\lambda_{j+n} - \lambda_j)} N_j^{n-1}(t) + \frac{(\lambda_{j+n+1} - t)}{(\lambda_{j+n+1} - \lambda_{j+1})} N_{j+1}^{n-1}(t)$$

where $N_j^0(t) = \begin{cases} 1 & for\ \lambda_j \le t < \lambda_{j+1} \\ 0 & otherwise \end{cases}$

and $N_j^n(t_i) = 0$ when t < λ_j or t > λ_{j+n+1}

For example, in the cubic case (i.e. n = 3) when $6 \le t < 8$ (i.e. $t_3 \le t < t_4$), it can be determined that j = 6, since j is found from the interval in the knot vector where $\lambda_j \le t < \lambda_{j+1}$, which for this case is $\lambda_6 \le t < \lambda_7$. From the definition of the recurrence equation above, and knowing that j=6 allows the recurrence equation to be initialized with $N_6^0(t) = 1$ and used to compute the B-splines of degree n= 1 to 3 that are nonzero over the interval $\lambda_6 \le t < \lambda_7$.

As above, for the case where $t_3 \le t < t_4$ (i.e. $\lambda_6 \le t < \lambda_7$) it follows that:

n= 0: $N_6^0(t) = 1$

$N_j^0(t) = 0$ for all other j (i.e. except $j = 6$)

n=1:
$$N_6^1(t) = \frac{(t-\lambda_6)}{(\lambda_7-\lambda_6)} N_6^0(t) + \frac{(\lambda_8 - t\ )}{(\lambda_8-\lambda_7)} N_7^0(t) = \frac{(t\ \ -\lambda_6)}{(\lambda_7-\lambda_6)} N_6^0(t)$$

$$N_5^1(t) = \frac{(t-\lambda_5)}{(\lambda_6-\lambda_5)} N_5^0(t) + \frac{(\lambda_7 - t)}{(\lambda_7-\lambda_6)} N_6^0(t_3) = \frac{(\lambda_7 - t)}{(\lambda_7-\lambda_6)} N_6^0(t)$$

$N_j^1(t) = 0$ for all other j (i.e. except j = 6 and 5)

n=2:
$$N_6^2(t) = \frac{(t-\lambda_6)}{(\lambda_8-\lambda_6)} N_6^1(t) + \frac{(\lambda_9 - t)}{(\lambda_9-\lambda_7)} N_7^1(t) = \frac{(t-\lambda_6)}{(\lambda_8-\lambda_6)} N_6^1(t)$$
$$N_5^2(t) = \frac{(t-\lambda_5)}{(\lambda_7-\lambda_5)} N_5^1(t) + \frac{(\lambda_8 - t)}{(\lambda_8-\lambda_6)} N_6^1(t)$$
$$N_4^2(t) = \frac{(t-\lambda_4)}{(\lambda_6-\lambda_4)} N_4^1(t) + \frac{(\lambda_7 - t)}{(\lambda_7-\lambda_5)} N_5^1(t) = \frac{(\lambda_7 - t)}{(\lambda_7-\lambda_5)} N_5^1(t)$$

$N_j^2(t) = 0$ for all other j (i.e. except j = 6, 5 and 4)

n=3:
$$N_6^3(t) = \frac{(t-\lambda_6)}{(\lambda_9-\lambda_6)} N_6^2(t) + \frac{(\lambda_{10} - t)}{(\lambda_{10}-\lambda_7)} N_7^2(t) = \frac{(t-\lambda_6)}{(\lambda_9-\lambda_6)} N_6^2(t)$$
$$N_5^3(t) = \frac{(t-\lambda_5)}{(\lambda_8-\lambda_5)} N_5^2(t) + \frac{(\lambda_9 - t)}{(\lambda_9-\lambda_6)} N_6^2(t)$$
$$N_4^3(t) = \frac{(t-\lambda_4)}{(\lambda_7-\lambda_4)} N_4^2(t) + \frac{(\lambda_8 - t)}{(\lambda_8-\lambda_5)} N_5^2(t)$$
$$N_3^3(t) = \frac{(t-\lambda_3)}{(\lambda_6-\lambda_3)} N_3^2(t) + \frac{(\lambda_7 - t)}{(\lambda_7-\lambda_4)} N_4^2(t) = \frac{(\lambda_7 - t)}{(\lambda_7-\lambda_4)} N_4^2(t)$$

$N_j^3(t) = 0$ for all other j (i.e. except j = 6, 5, 4 and 3)

Step 3. Compute the 2nd derivatives of the Bspline basis functions, $N_j^n(t)$, at t_0 and t_m using the recurrence equation,

$$^{(\ell)}N_j^n(t) = n\left(\frac{1}{(\lambda_{j+n}-\lambda_j)}\,^{(\ell-1)}N_j^{n-1}(t) \quad - \quad \frac{1}{(\lambda_{j+n+1}-\lambda_{j+1})}\,^{(\ell-1)}N_{j+1}^{n-1}(t)\right)$$

$$\text{where } N_j^0(t) = \begin{cases} 1 & for\ \lambda_j \le t < \lambda_{j+1} \\ 0 & otherwise \end{cases}$$

and $\ell = \ell^{th}$ derivative

First compute $N_j^n(t_0)$, n = 0 to 3 where j = 0 to 3 using the standard Bspline recurrence equation. It turns out that when $t = t_0$, $N_3^n(t_0) = 0$ for n = 1 to 3. With knowledge of the values of $N_j^n(t_0)$ in hand, the first derivative of $N_j^n(t_0)$ can be computed as follows:

First derivative of n=2 Bspline basis functions at t_0 (i.e. case $n = 2, \ell = 1$):

$$^{(1)}N_3^2(t_0) = 2\left(\frac{1}{(\lambda_5-\lambda_3)}N_3^1(t_0) \quad - \quad \frac{1}{(\lambda_6-\lambda_4)}N_4^1(t_0)\right) = \frac{2}{(\lambda_5-\lambda_3)}N_3^1(t_0) = 0$$

$$^{(1)}N_2^2(t_0) = 2\left(\frac{1}{(\lambda_4-\lambda_2)}N_2^1(t_0) \quad - \quad \frac{1}{(\lambda_5-\lambda_3)}N_3^1(t_0)\right) = \frac{2}{(\lambda_4-\lambda_2)}N_2^1(t_0)$$

$$^{(1)}N_1^2(t_0) = 2\left(\frac{1}{(\lambda_3-\lambda_1)}N_1^1(t_0) \quad - \quad \frac{1}{(\lambda_4-\lambda_2)}N_2^1(t_0)\right) = \frac{-2}{(\lambda_4-\lambda_2)}N_2^1(t_0)$$

Knowing $^{(1)}N_3^2(t_0)$, $^{(1)}N_2^2(t_0)$, $^{(1)}N_1^2(t_0)$ enables the second derivative of the n=3 (i.e. cubic) B-splines to be computed as follows:

$$n = 3, \ell = 2: \quad ^{(2)}N_3^3(t_0) = 3\left(\frac{1}{(\lambda_6-\lambda_3)}\,^{(1)}N_3^2(t_0) \quad - \quad \frac{1}{(\lambda_7-\lambda_4)}\,^{(1)}N_4^2(t_0)\right) = 0$$

$$^{(2)}N_2^3(t_0) = 3\left(\frac{1}{(\lambda_5-\lambda_2)}\,^{(1)}N_2^2(t_0) \quad - \quad \frac{1}{(\lambda_6-\lambda_3)}\,^{(1)}N_3^2(t_0)\right)$$

$$^{(2)}N_1^3(t_0) = 3\left(\frac{1}{(\lambda_4-\lambda_1)}\,^{(1)}N_1^2(t_0) \quad - \quad \frac{1}{(\lambda_5-\lambda_2)}\,^{(1)}N_2^2(t_0)\right)$$

$$^{(2)}N_0^3(t_0) = 3\left(\frac{1}{(\lambda_3-\lambda_0)}\,^{(1)}N_0^2(t_0) \quad - \quad \frac{1}{(\lambda_4-\lambda_1)}\,^{(1)}N_1^2(t_0)\right)$$

The values of $^{(2)}N_j^n(t_m)$ can be computed in a similar fashion.

Step 4. Set up the vector matrix equation of the form,

$$\mathbf{Ac} = \mathbf{d}$$

and solve for the coefficient vector $\mathbf{c} = \mathbf{A}^{-1}\mathbf{d}$ where,

- **A** is a matrix of dimension $(m+n)$ x $(m+n)$ containing the Bspline basis function values (and derivatives) evaluated at each t_i, $i = 0$ to m,
- **c** is a coefficient vector of dimension $(m+n)$ x 1, and
- **d** is a vector of dimension $(m+n)$ x 1 containing the desired values of the spline at the points p_i, i = 0 to m (and possibly 1st or 2nd derivative values at points t_0 and t_m).

For the cubic case (n = 3) with natural spline endpoint conditions (i.e. second derivative = 0 at t_0 and t_m),

the matrix **A** takes the banded form (where all entries other than the ones shown are assumed to be zero):

$$
\mathbf{A} = \begin{bmatrix}
{}^{(2)}N_0^3(t_0) & {}^{(2)}N_1^3(t_0) & {}^{(2)}N_2^3(t_0) & {}^{(2)}N_3^3(t_0) & & & & & & \\
N_0^3(t_0) & N_1^3(t_0) & N_2^3(t_0) & N_3^3(t_0) & & & & & & \\
 & N_1^3(t_1) & N_2^3(t_1) & N_3^3(t_1) & N_4^3(t_1) & & & & & \\
 & & N_2^3(t_2) & N_3^3(t_2) & N_4^3(t_2) & N_5^3(t_2) & & & & \\
 & & & \ddots & \ddots & \ddots & & & & \\
 & & & & N_{m-3}^3(t_{m-3}) & N_{m-2}^3(t_{m-3}) & N_{m-1}^3(t_{m-3}) & N_m^3(t_{m-3}) & & \\
 & & & & & N_{m-2}^3(t_{m-2}) & N_{m-1}^3(t_{m-2}) & N_m^3(t_{m-2}) & N_{m+1}^3(t_{m-2}) & \\
 & & & & & & N_{m-1}^3(t_{m-1}) & N_m^3(t_{m-1}) & N_{m+1}^3(t_{m-1}) & N_{m+2}^3(t_{m-1}) \\
 & & & & & & N_{m-1}^3(t_m) & N_m^3(t_m) & N_{m+1}^3(t_m) & N_{m+2}^3(t_m) \\
 & & & & & & {}^{(2)}N_{m-1}^3(t_m) & {}^{(2)}N_m^3(t_m) & {}^{(2)}N_{m+1}^3(t_m) & {}^{(2)}N_{m+2}^3(t_m)
\end{bmatrix}
$$

$\Rightarrow$ when n = 3 the matrix **A** has (m+3) rows and (m+3) columns)

The vector **d** takes the form,

$$\mathbf{d} = \begin{bmatrix} 0 & p_0 & p_1 & \dots & p_m & 0 \end{bmatrix}^T$$

and the coefficient vector **c** is determined from the equation,

$\mathbf{c} = \mathbf{A}^{-1}\mathbf{d}$ where $\mathbf{A}^{-1}$ is the inverse of the matrix **A.**

For example, given the seven points (t_i, p_i), i=0 to 6 shown on the figure of the first page with values,

$$p_0=(0,1),\ p_2=(2,2),\ p_3=(4,2.5),\ p_4=(6,1.5),\ p_5=(8,0.5),\ p_6=(10,0.25)$$

and the associated knot array,

$$\begin{aligned}\boldsymbol{\lambda} &= [-6 \quad -4 \quad -2 \ \vdots\ 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \ \vdots\ 14 \quad 16 \quad 18] \\ &= [\lambda_0 \quad \lambda_1 \quad \lambda_2 \quad \ldots \quad \lambda_{12}]\end{aligned}$$

it follows that,

$$\mathbf{d} = [0 \ \vdots\ 1 \quad 2 \quad 2.5 \quad 1.5 \quad 0.5 \quad .25 \quad 0 \ \vdots\ 0]^T$$

$$\mathbf{A} = \begin{bmatrix} 0.25 & -0.5 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0.167 & 0.667 & 0.167 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.167 & 0.667 & 0.167 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.167 & 0.667 & 0.167 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.167 & 0.667 & 0.167 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.167 & 0.667 & 0.167 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.167 & 0.667 & 0.167 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.167 & 0.667 & 0.167 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0.25 & -0.5 & 0.25 \end{bmatrix}$$

and $\mathbf{c} = \mathbf{A}^{-1}\mathbf{d} = [-0.03 \quad 1.0 \quad 2.0301 \quad 2.8795 \quad 1.4519 \quad 0.3128 \quad 0.2968 \quad 0 \quad -0.297]^T$

Computing the Interpolating Bspline function f(t):

Since there are only $n+1$ nonzero basis splines for any value of t, the interpolating Bspline function $f(t) = \sum_{j=0}^{m+n-1} c_j N_j^n(t)$ can be simplified to,

$$f(t) = \sum_{k=0}^{n} c_{j-k} N_{j-k}^n(t)$$

where j is associated with the interval in which $\lambda_j \le t < \lambda_{j+1}$

Step 1. Given the knot vector $\boldsymbol{\lambda}$ and a value for t find the value of j such that $\lambda_j \le t < \lambda_{j+1}$

Step 2. Given the value of j determined in Step 1 and the value of t, compute the (n+1) Bspline basis functions $N_j^n(t)$ that are nonzero over the interval $\lambda_j \le t < \lambda_{j+1}$ using the Bspline recurrence equation described previously. For example, when t=7 the nonzero Bspline basis functions $N_j^n(t)$ up to degree n=3 that are nonzero over the interval $\lambda_6 \le t < \lambda_7$ (i.e. $6 \le t < 8$) would be found to be,

$$\begin{array}{cccc}
 & & & N_6^0 = 1 \\
 & & N_5^1 = \frac{1}{2} & N_6^1 = \frac{1}{2} \\
 & N_4^2 = \frac{1}{8} & N_5^2 = \frac{3}{4} & N_6^2 = \frac{1}{1} \\
N_3^3 = \frac{1}{48} & N_4^3 = \frac{23}{48} & N_5^3 = \frac{23}{48} & N_6^3 = \frac{1}{48}
\end{array}$$

Step 3. Assuming the coefficient vector **c** has been previously determined, for the cubic case (n=3) the Bpline function $f(t)$ can be evaluated at t=7 as,

$$f(t) = c_{j-3} N_{j-3}^3(t) + c_{j-2} N_{j-2}^3(t) + c_{j-1} N_{j-1}^3(t) + c_j N_j^3(t)$$

where j is the index value determined in Step 2.

As shown in the example of Step 2 when t=7, j = 6 so it follows that,

$$f(t) = c_3 N_3^3(t) + c_4 N_4^3(t) + c_5 N_5^3(t) + c_6 N_6^3(t)$$

where values for the basis splines $N_3^3(t=7) = \frac{1}{48}$, $N_4^3(t=7) = \frac{23}{48}$, $N_5^3(t=7) = \frac{23}{48}$, $N_6^3(t=7) = \frac{1}{48}$ at t=7 are obtained as shown in Step 2.